PT00309

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THE TRAJECTORY SIMULATION PROGRAMS OF THE PHYSICAL SCIENCE LABORATORY OF NEW MEXICO STATE UNIVERSITY

NASA CR70733

KEITH GUARD

30 March 1962

Prepared For:
National Aeronautics & Space Administration
Goddard Space Flight Center
Contract NAS 5-1032
and
Bell Telephone Laboratories
Contract No. POD-232136

NEW MEXICO STATE UNIVERSITY
PHYSICAL SCIENCE LABORATORY
UNIVERSITY PARK, NEW MEXICO

S/34987

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By
Physical Science Laboratory
New Mexico State University
University Park, New Mexico

ABSTRACT

The rocket trajectory simulation programs in use at the Physical Science Laboratory of New Mexico State University are described. The purpose and methods of use of the programs are discussed, the theory is outlined, and the equations are listed in summary form.

THE TRAJECTORY SIMULATION PROGRAMS

OF PSL/NMSU

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THE TRAJECTORY SIMULATION PROGRAMS
OF PSL/NMSU

1. Purpose of the Programs

1.1 The trajectory simulation programs now in use by the Physical Science Laboratory of New Mexico State University are designed to simulate multi-stage, unguided rocket trajectories from launch to impact. The programs were designed to supplement each other to furnish data for the following general objectives:

(a) Investigate performance characteristics such as peak altitude and range to impact for a given rocket configuration as functions of launch angle and payload.

(b) Obtain the necessary tables for use in computing corrections to launcher setting to compensate for wind effects.

(c) Compute estimates of impact dispersion.

1.2 A trajectory simulation program is a special case of the time domain solution, by iterative methods, of a set of ordinary differential equations starting with given initial conditions.

From the digital computer programmer's point of view, a "rocket" is a set of n differential equations, which can be written

$$\dot{x}_i = f_i\ (x_j,\ t) \qquad i,j = (1,\ 2,\ .\ .\ .,\ n) \qquad \text{1-1}$$

and for which a set of initial conditions x_{io}, t_o are known. The functions

f_i are, in general, non-linear and in fact contain functional relationships which are expressed only in tabular form. For equations of this type, it is impossible to write a closed form solution of 1-1. That is, it is impossible to find functions ϕ_i such that

$$x_i \, (t) = \phi_i \, (x_{jo}, \, t_o). \qquad \text{1-2}$$

If the functions f_i are such that x_i can be expressed by a Taylor's series expansion about t_o, the equations (1-1) can be integrated by a numerical technique over a small integration interval Δt. If Δt is chosen small enough, $x_i \, (t_o + \Delta t)$ can be evaluated to any desired accuracy. The particular method used in the PSL trajectory program is given in 2.1 and is derived in Reference 1.

The output values of the first step integration program, $x_i \, (t_o + \Delta t)$, $(t_o + \Delta t)$ are used for initial conditions to compute the next point, and so on.

Use of this method means that computation of impact coordinates for a given set of initial conditions will require the step-by-step integration of the entire trajectory from launch to impact. Each change in initial conditions, and each change in the functional description of the rocket will require a complete new trajectory computation.

Since this process is extremely time consuming, every possible means is used to shorten the computations.

1.3 The use of several programs is motivated by the fact that each program, when designed for a specific purpose, is more efficient than a single general purpose program would be.

Five programs are in use:

<u>Program</u>

1	Two-Dimensional Particle Trajectory
2	Two-Dimensional Rigid Body Standard Trajectory
3	Two-Dimensional Rigid Body with Wind Trajectory
4	Two-Dimensional Rigid Body with Malalignments Trajectory
5	Three-Dimensional Particle with Curved Earth Trajectory

<u>1.3.1</u> All five programs have certain features in common, including:

(a) Tables and equations for thrust, mass and drag.

(b) Atmospheric data.

(c) Table look-up routine.

(d) Integration routine.

(e) Time interval control, and discontinuity check.

These features are discussed in Section 2.

<u>1.3.2</u> The <u>limitations</u> common to all five programs are:

(a) Wind effects and dispersion terms can be computed only in the pitch plane. The effects normal to the flight path, which are variously termed yaw effects, heading effects or cross-range effects, must be estimated from the pitch plane data.

(b) Coupling effects in yaw and pitch caused by roll cannot be computed.

(c) Aerodynamic terms are limited to linearized coefficients.

(d) Gyroscopic effects caused by missile roll cannot be computed.

1.4 The Two-Dimensional Particle Trajectory Program was designed to obtain rocket performance characteristics as functions of launch angle and payload. It can also be used to compute performance (altitude and range) variations caused by

(a) Thrust variation, expressed as a percentage of normal thrust vs. time.

(b) Drag variation, expressed as a percentage variation of the drag coefficient as a function of Mach Number.

(c) Second stage ignition time variation, or interstage delay time.

1.4.1 The particle trajectory equations constrain the thrust and drag forces to act along the velocity vector. The equations can also be described as "zero angle of attack", "infinite stability", or "zero inertia" equations. The equations cannot describe:

(a) Wind effects

(b) Aerodynamic or thrust malalignment effects

(c) Vehicle response lag to any change of conditions.

The response lag limitation causes the trajectory description to be inaccurate for zero-length or low velocity launches. If the launch velocity is relatively high (over 150 ft/sec.) the trajectories as computed correspond very closely to those obtained from the rigid body equations with no perturbations. The particle trajectory is used for overall performance evaluation, wherever possible, because of:

(a) Much greater speed of computing,

(b) Relative simplicity of set-up for computation.

1.4.2 The force equations contain:

(a) Gravitational force directed parallel to the vertical (z) coordinate and varying inversely as the square of the distance from the earth's center.

(b) Thrust force, acting along the velocity vector.

(c) Drag force, acting opposite to the velocity vector.

1.5 Two-Dimensional Rigid Body Standard Trajectory Program was designed to:

(a) Provide performance and performance variation data when response lag characteristics do not allow the Particle Trajectory Program to be used.

(b) Provide a standard for computing performance variation when perturbations are introduced in the other rigid body programs. For example, the difference between impact range for a given launch angle and a specified wind, as computed by Program 3, and the impact range for the same rocket at the same launch angle, as computed by this program, gives the impact displacement due to the specified wind. The equations and tables of this program are the same as those of the other two, except for the terms for wind effects in Program 3 and the terms for malalignment effects in Program 4.

(c) Compute performance variation and dispersion caused by variations in pitch angle and pitch angular velocities at the instant of separation from the launcher. The variations are usually called "tip-off".

1.5.1 The forces and moments acting on the rocket are:

(a) Gravitational force as used in Program 1.

(b) Thrust vector magnitude as used in Program 1, acting along the longitudinal axis of the rocket.

(c) Drag force as used in Program 1.

(d) Lift force, acting normal to the drag force, and proportional to the sine of twice the angle of attack.

(e) Aerodynamic restoring moment produced by the resultant of lift and drag forces, acting at the center of pressure.

(f) Aerodynamic damping moment.

(g) Jet damping moment.

1.6 The Two-Dimensional Rigid Body with Wind Trajectory was designed to compute:

(a) Effect on range of a uniform wind acting on the rocket from launch altitude to an altitude of negligible wind effect. Range effect of a unit wind is called the unit wind effect, and is considered to be a function of launch angle.

(b) Wind weighting function "f" as a function of altitude.

(c) Effect on the trajectory of any wind condition which can be represented as a tabular function of altitude.

1.6.1 The equations for forces and moments acting on the rocket are the same as those of Program 2, except:

(a) Drag force acts in the direction of the relative velocity vector, which is the velocity vector of the rocket relative to the moving air.

(b) Lift force is normal to the relative velocity vector.

1.7 The Two-Dimensional Rigid Body with Malalignments Trajectory Program was designed to provide:

(a) Computation of the effect on the trajectory of a thrust malalignment on each stage independently.

(b) Computation of the effect on the trajectory of a fin malalignment on each stage independently.

1.7.1 In a two-dimensional program, malalignment effects are computed in the pitch plane only.

The equations compute this effect by:

(a) Integrating the roll rate and adding an initial roll angle to obtain roll angle.

(b) Multiplying the malalignment forces and moments by the sine of the roll angle to obtain the pitch plane component of the forces and moments.

1.7.2 The equations for forces and moments are the same as in Program 2, with the addition of:

(a) A thrust malalignment force component, acting normal to the rocket longitudinal axis; equal to the thrust force, as used in Programs 1 through 3, multiplied by the sine of a malalignment angle and by the sine of the roll angle.

(b) A thrust malalignment moment, obtained by multiplying the thrust malalignment force by the distance from the rocket center of mass to the rocket motor throat.

(c) A fin malalignment force component, acting normal to the relative velocity vector; equal to the fin lift caused by the malalignment angle, multiplied by the sine of the roll angle.

(d) A fin malalignment moment, obtained by multiplying the fin malalignment force by the cosine of the angle of attack, to obtain the component normal to the rocket axis, then multiplying by the distance from the rocket center of mass to the center of pressure of the fins.

1.8 The calculations using the rigid body equations take much more time than those using the particle equations. The equations are more complex, and a shorter integration step interval must be used. Where the particle equation computation may be stable and sufficiently accurate with an interval of one second, the rigid body equation computations require integration intervals as small as .01 or .02 seconds to maintain stability and accuracy.

To save time, in Programs 2, 3, and 4, when the rocket oscillations damp out so that the angle of attack is negligible, the equations are automatically changed from rigid body equations to particle equations.

1.9 In Programs 1 through 4, the motion can be computed by a closed form solution when the rocket is in vacuum (above 300,000 feet) and has no thrust. The only force in this case is an inverse square gravity. The two points computed in this manner are peak and atmospheric re-entry at 300,000 feet on the descent. The closed form solution is optional and is used for time saving when detailed trajectory above 300,000 feet is not wanted.

1.10 The Three-Dimensional Particle Trajectory with Curved Earth Program contains equations for description of:

(a) The earth's surface, which is described as an ellipsoid of revolution.

(b) The earth's gravitational field which is described by an expansion in terms of zonal harmonics up to the sixth harmonic.

(c) The effects of the earth's rotation; the Coriolis and centrifugal force terms.

Tesseral harmonics (variations with longitude) and the effects of local irregularities of the earth's surface are not included in the equations. For this reason the gravitational force as computed for any specific point on the earth's surface may not agree exactly with the force as measured at that point.

1.10.1 The Curved Earth Program was designed to provide:

(a) Impact displacement caused by Coriolis force effects.

(b) Height above the earth's surface, rather than a Cartesian vertical coordinate with respect to the launcher.

(c) Impact locations on the curved earth surface, rather than on a Cartesian plane tangent to the earth at the launcher.

(d) An accurate description of long range free flight rocket trajectories.

1.10.2 The force equations contain:

(a) Gravitational and rotational forces as discussed in 1.10 above.

(b) Thrust force, acting along the velocity vector in three dimensions.

(c) Drag force, acting opposite to the velocity vector.

2. The General Program

The five programs under discussion are special cases of a general trajectory program. The rocket equations in each of the five are used as sub-routines for the general program. The efficiency and utility of any trajectory simulation are established by the logic of the general program; and the programming of this logic is the most critical task in writing a simulation program.

The major components of the general program are described below.

2.1 The integration routine used is the Runge-Kutta fourth order method* in which, given a set of n differential equations

$$\dot{x}_i = f_i\ (x_j,\ t) \qquad i,\ j = 1,\ 2,\ 3,\ .\ .\ .,\ n$$

with the initial conditions

$$t_o,\ x_{io},$$

the solution at the end of a time interval Δt is computed by the equations

$$k_{i1} = \Delta t\ f_i\ (x_{jo},\ t_o)$$

$$k_{i2} = \Delta t\ f_i\ \left[(x_{jo} + 1/2\ k_{j1}),\ (t_o + 1/2\ \Delta t)\right]$$

$$k_{i3} = \Delta t\ f_i\ \left[(x_{jo} + 1/2\ k_{j2}),\ (t_o + 1/2\ \Delta t)\right]$$

*Reference 1.

$$k_{i3} = \Delta t \; f_i \left[(x_{jo} + k_{j3}), \; (t_o + \Delta t) \right]$$

$$x_i \; (t_o + \Delta t) = x_{io} + (1/6)(k_{i1} + 2k_{i2} + 2k_{i3} + k_{i4}).$$

2.2 The integration routine has a provision for testing to find whether the Δt used is adequate. This test is made every q iterations, where q is a constant which can be set in advance. The test is as follows:

(a) The interval Δt is cut in half, two iterations are computed and the results stored.

(b) The interval Δt is restored to its original value, one iteration is computed with the original starting time, and the results stored.

(c) For each variable x_i being integrated, the value obtained from (a) is subtracted from the value obtained in (b), and the difference divided by the value obtained in (a).

(d) Each quotient so obtained is the relative error in the x_i, due to use of the time interval Δt. Each quotient is compared against a tolerance T, and if any quotient is greater than T, the time interval for integration is set at $\Delta t/2$, and the program proceeds. The tolerance T is a constant of the program which can be set in advance.

(e) If all quotients obtained in (c) are less than T, another iteration is performed using the time interval Δt, and the results stored.

(f) The time interval is set at $2\Delta t$, and one iteration with the original initial condition is performed and the results stored.

(g) For each variable x_i, a relative error is obtained between the values obtained in (e) and those obtained in (f), and each of these errors compared to T.

(h) If any relative error is greater than T, the time interval is set at Δt, and the program proceeds.

(i) If no relative error is greater than T, the time interval is set at $2\Delta t$ and the program proceeds.

The preceding computation is necessary to insure the time interval used will not be too small for efficient computer use, and not too large for stability and accuracy.

2.3 The time discontinuity control in the general program is governed by a table of times of functional discontinuities including

(1) Stage separations

(2) Rocket motor burnouts and ignitions

(3) Abrupt changes in slope of the thrust curve

(4) Payload separations.

All time function tables contain double values at discontinuities. Before each integration step, the program tests whether the integration interval will contain a discontinuity time. If not, the program proceeds, using the same Δt. If the interval contains a discontinuity, Δt is reduced to a value which makes the end of the interval coincide with the discontinuity. The first set of the double tabular values are used to compute the function values at the end of the interval.

For the next computation after a discontinuity the time interval is automatically reset to a programmed value and the second set of the double tabular values are used to compute the function values at the start of the interval.

The program also automatically selects the appropriate rocket function tables to use after each discontinuity.

2.4 The altitude discontinuity control in the general program includes a table of altitudes of functional discontinuities including

(1) Launcher exit

(2) Wind strata boundaries

(3) Atmosphere boundary (300,000 feet)

(4) Impact.

At the end of each integration step, the program tests whether the integration has included an altitude discontinuity. If not, the program proceeds. If so, linear interpolation of time vs. altitude is performed to obtain a new value of Δt, and the integration step repeated. Since time vs. altitude is generally not linear, the recomputed altitude will not fall exactly on the discontinuity. The process of testing, interpolation and recomputing is repeated until the altitude at the end of the interval is within a small preset tolerance limit of the discontinuity. The action taken depends on the nature of the discontinuity.

(1) At launcher exit, the rocket equations of motion are changed from launcher-constrained to free-flight equations.

(2) At wind strata boundaries, the wind function values are changed.

(3) At atmospheric exit, the equations are changed to those for motion in a vacuum; and at re-entry, they are changed back.

(4) At impact the program halts, initial conditions for the next trajectory are established and the program started.

2.5 The general program contains a table look-up routine and storage for all tabular values of rocket functions. The format for tables is the same for all programs. The table look-up is a linear interpolation.

The functions of time (thrust, mass, center of gravity and moment of inertia) are tabulated sequentially for the entire trajectory, with double entries for discontinuities.

The table containing thrust vs. time is a multiple entry table, listing in separate columns:

(a) sea level thrust

(b) exit nozzle area (zero when thrust is zero)

(c) time of next discontinuity

(d) code number (1 through 4) to indicate which aerodynamic tables are applicable.

The aerodynamic tables are functions of Mach Number and include aerodynamic coefficients, reference areas, and centers of pressure. Each table is a multiple entry table, containing four sets of values of the dependent variable. Each set of values applies to one phase of the rocket trajectory. For example, for a two stage rocket three sets are used--boost phase, second stage coast and second stage burning. Selection of the phase is a time function, controlled by the code number in the thrust table.

Atmospheric functions are tabulated vs. altitude. The functions, speed of sound and atmospheric pressure, are condensed from Tables IV-A and IV-D of Reference 2.

3. Employment of the Programs

The programs are used for obtaining a performance evaluation, wind functions, and/or dispersion analysis.

3.1 Performance evaluation is the basic use of Programs 1, 2, and 5, and can be used to:

(a) Study feasibility of rocket design.

(b) Test sensitivity of performance to changes in configuration.

(c) Provide parameters for other studies including heat transfer and instrumentation environment.

(d) Provide a standard for wind and dispersion studies.

(e) Provide knowledge of the probable flight path of a rocket.

For performance evaluation Program 1, which requires a small number of input parameters and less computation time, is the most economical. Program 2, however, provides rigid body data and more performance characteristics if valid input is available. Program 2 must be used initially if launch is from a zero-length launcher.

The performance of a particular rocket is considered a function of its payload, launch angle and flight time, and is characterized by parameters including range, altitude and velocity. A description of a rocket's performance is obtained by choosing values of these parameters at selected flight events (burnout, peak, impact, etc.) for various launch angles and payloads.

The performance characteristics are converted to the proper units and tabulated against suitable variables. Common performance tables are: Range to Impact vs. Launch Angle and Range to Impact vs. Payload.

3.2 The wind functions, wind weighting function and unit wind effect, are obtained from Programs 2 and 3.

The wind weighting function, f (z), is obtained by computing trajectories (Program 3) through wind layers of increasing height, z_w, up to a limit of usually 100,000 feet altitude. A constant wind, V_w, acts within each layer. The computing technique does not run all the wind layer trajectories from launch, since this would be costly duplication of segments of trajectories. The trajectory is computed to the top of the first wind layer and initial conditions are stored, then the trajectory is computed to impact. The computer then takes the stored initial conditions and computes to the top of the next layer, stores new initial conditions and computes to impact. The process iterates until the last wind layer top is reached. The resulting impact ranges are compared to the Program 2 impact range and a wind weight value is calculated for each wind layer heighth after corrections for drift.

Wind Weighting Function (f (z)) =

$$\frac{\text{Displacement of impact due to wind to alt. z}}{\text{Displacement of impact due to wind to alt. 100,000 ft.}}$$

Experience has shown the wind weighting function to be nearly independent of launch angle and payload, so usually one set of wind layer trajectories at a nominal launch angle and payload will be sufficient for each rocket.

The unit wind effect, $\delta(\theta)$, is obtained by computing trajectories (Program 3) with wind to maximum altitude (usually 100,000 feet) over a range of launch angles, and comparing impacts obtained with those from Program 2.

$$\delta(\theta) = (\text{Range to impact with } \underline{\text{unit wind}} \text{ to maximum altitude})$$
$$- (\text{Range to impact with } \underline{\text{no wind}}).$$

The unit wind effect varies with payload, making it necessary to compute $\delta(\theta)$ for at least the maximum, minimum, and nominal payloads.

Actual wind profiles can also be used in Program 3 to obtain impact predictions and study rocket response to high velocity wind and wind shears.

3.3 The dispersion of unguided rockets due to atmospheric effects and deviations from design criteria is estimated using Programs 1, 2, 3 and 4. The perturbing influences are listed below in "rough order" of decreasing effect:

(a) Thrust malalignment on each stage
(b) Wind uncertainty
(c) "Tip-off" (initial pitch rate at separation from tower)
(d) Launch angle uncertainty
(e) Second stage ignition time variation
(f) Thrust variation
(g) Drag variation
(h) Payload uncertainty
(i) Fin malalignment on each stage.

The tabulated factors make the largest contributions to dispersion. Perturbations caused by center of thrust, coefficient of lift, and center of pressure variations have small effect on the trajectory and are not usually considered in dispersion studies.

Each dispersion factor is varied independently. The unit range effect due to each factor is multiplied by the estimated probable deviation to give estimated dispersion. Two estimated dispersion tabulations are derived, one for range and one for cross-range. Where the perturbation is known to have cross-range effect, the magnitude is assumed equal to the range effect.

The total dispersion is calculated for range and cross-range by finding the square root of the sum of the squares of the estimated dispersions.

3.4 The operational flow insuring the most economical route to obtaining the desired performance evaluation, wind functions, and/or dispersion analysis is shown in Appendix III.

Three basic tasks must be completed before any simulation computation can begin:

(a) The simulation plan, which outlines the number and types of computer simulations necessary for the planned operation, must be approved by the client.

(b) The client must supply the input parameters necessary for the operation or approve applicable parameters that PSL has available. A list of required input parameters and tables is included in Appendix II.

(c) The client must supply the estimated probable deviation (3σ values) of the dispersion factors.

4. Force and Moment Equations

The vector and matrix notation used throughout the balance of this report is defined in Section 9.

4.1 Newton's Second Law applied to a rigid body is

$$\frac{d}{dt}(M\vec{\dot{R}}^{I}) = \Sigma \vec{F}^{I} \qquad 4\text{-}1$$

M	Mass of the body
$\vec{\dot{R}}^{I}$	Velocity of the body in an inertial coordinate system
$\Sigma\vec{F}^{I}$	Sum of the external forces acting on the body, resolved in the same inertial system.

If the reaction effect of the rocket motor is written as one of the forces, equation (4-1) as applied to a rocket is*

$$M\vec{\ddot{R}}_m{}^{I} = M\vec{G}^{I} + \vec{T}^{I} + \vec{A}^{I} \qquad 4\text{-}2$$

$\vec{\ddot{R}}_m{}^{I}$	Inertial acceleration of the rocket
$M\vec{G}$	Gravitational force
$\vec{T}$	Thrust or reaction force

*Reference 3, Chapter I.

$\vec{A}$ Aerodynamic force

If g_i is a ground fixed system and $\left[c_{ij}\right]$ is the transformation matrix between the inertial system and g_i,

$$\vec{R}_m{}^g = \left[c_{ij}\right] \vec{R}_m{}^I + \vec{R}_I{}^g$$

as defined in 9.2.3. From (9-46), with $\dot{\omega}_i = 0$, the acceleration in the ground system is

$$\ddot{\vec{R}}_m{}^g = \left[c_{ij}\right] \ddot{\vec{R}}_m{}^I - 2\,\vec{\omega}^g \times \vec{R}_m{}^g \times \vec{\omega}^g \times \vec{\omega}^g \times \left[c_{ij}\right] \vec{R}_m{}^I$$

$$= \left[c_{ij}\right] \ddot{\vec{R}}_m{}^I + \vec{\mathcal{C}} + \vec{C}$$

$\vec{\omega}$ Angular velocity of the earth

$\vec{\mathcal{C}}$ Coriolis acceleration

$\vec{C}$ Centrifugal acceleration

$$\vec{\mathcal{C}}^g = -2\,\vec{\omega}^g \times \vec{R}_m{}^g$$

$$\vec{C}^g = -\,\vec{\omega}^g \times \vec{\omega}^g \times \left[c_{ij}\right] \vec{R}_m{}^I .$$

Substituting (4-2) into (4-3) yields

$$\ddot{\vec{R}}_m{}^g = \frac{1}{M} \left[c_{ij}\right] (\vec{MG}^I + \vec{T}^I + \vec{A}^I) + \vec{\mathcal{C}}^g + \vec{C}^g$$

$$= \frac{1}{M} (\vec{T}^g + \vec{A}^g) + \vec{G}^g + \vec{\mathcal{C}}^g + \vec{C}^g . \qquad 4\text{-}4$$

4.2 Equation (4-4) is the form of the force equation used in the particle trajectory programs. The ground fixed coordinate system used is the ℓ_i system defined in 8.6. Equation (4-4) becomes

$$\ddot{\vec{R}}_m^{\ell} = \frac{1}{M}\left(\vec{T}^{\ell} + \vec{A}^{\ell}\right) + \vec{G}^{\ell} + \mathcal{C}^{\ell} + \vec{C}^{\ell}. \qquad 4\text{-}5$$

With the particle assumption of zero angle of attack:

$$\vec{T}^{\ell} = |T| \, \dot{\vec{r}}_m^{\ell} \qquad 4\text{-}6$$

$$\vec{A}^{\ell} = -|A| \, \dot{\vec{r}}_m^{\ell}. \qquad 4\text{-}7$$

As defined in 9.1.3 $\dot{\vec{r}}_m^{\ell}$ is the unit vector in the direction of $\dot{\vec{R}}_m^{\ell}$. The magnitude $|T|$ is discussed in Section 5 and the magnitude $|A|$ in Section 6. The vectors $\vec{G}$, $\vec{\mathcal{C}}$, $\vec{C}$ are lumped into an earth force vector, $\vec{E}^{\ell}$

$$\vec{E}^{\ell} = \vec{G}^{\ell} + \vec{\mathcal{C}}^{\ell} + \vec{C}^{\ell}. \qquad 4\text{-}8$$

In Programs 1 through 4,

$$\vec{E}^{\ell} = -g_o \left(\frac{R_o}{R_o + z_m^{\ell}}\right)^2 \vec{\ell}_3 \qquad 4\text{-}9$$

g_o Net acceleration due to $\vec{G} + \vec{C}$, on a stationary particle at the launcher

R_o Geocentric radius vector to the launcher

z_m^{ℓ} Height of rocket above launcher (paragraph 9.1.2).

The vector $\vec{E}^{\ell}$ for Program 5 is derived in Section 7.

4.3 In the rigid body trajectory equations, the forces are resolved in a body fixed coordinate system b_i (Figure 4-1). The equation relating body coordinates to the ℓ_i coordinates is

$$\vec{R}_m^{b} = \left[a_{ij}\right] \vec{R}_m^{\ell}. \qquad 4\text{-}10$$

The velocity relationship is

$$\dot{\vec{R}}_m{}^b = \left[a_{ij}\right] \dot{\vec{R}}_m{}^{\ell} - \vec{\omega}^b \times \vec{R}_m{}^b$$

which can be written

$$\left[a_{ij}\right] \dot{\vec{R}}_m{}^{\ell} = \dot{\vec{R}}_m{}^b + \vec{\omega}^b \times \vec{R}_m{}^b.$$

A vector $\vec{V}^b$ is defined:

$$\vec{V}^b = \left[a_{ij}\right] \dot{\vec{R}}_m{}^{\ell}. \qquad 4\text{-}11$$

$\vec{V}^b$ is the launcher system velocity vector rotated into the body system. It is <u>not</u> the body system velocity $\dot{\vec{R}}^b$. However, it is a perfectly legitimate vector, expressed in the body system, and its components can be differentiated in the b_i system. From (9-18) and (9-38)

$$\dot{\vec{V}}^b = \left[a_{ij}\right] \ddot{\vec{R}}_m{}^{\ell} + \left[\dot{a}_{ij}\right] \dot{\vec{R}}_m{}^{\ell} \qquad 4\text{-}12$$

$$= \left[a_{ij}\right] \ddot{\vec{R}}_m{}^{\ell} + \left[\omega_i{}^b\right] \left[a_{ij}\right] \dot{\vec{R}}_m{}^{\ell}.$$

Using (4-11)

$$\dot{\vec{V}}^b = \left[a_{ij}\right] \ddot{\vec{R}}_m{}^{\ell} - \omega^b \times \vec{V}^b. \qquad 4\text{-}13$$

From (4-5) and (4-10)

$$\left[a_{ij}\right] \ddot{\vec{R}}_m{}^{\ell} = \frac{1}{M} (\vec{T}^b + \vec{A}^b) + \vec{E}^b \qquad 4\text{-}14$$

$$\dot{\vec{V}}^b = \frac{1}{M} (\vec{T}^b + \vec{A}^b) + \vec{E}^b - \vec{\omega}^b \times \vec{V}^b. \qquad 4\text{-}15$$

This is the equation to be used in computation. The forces will be resolved into components in the body system; $\dot{\vec{V}}^b$ computed by equation (4-15) and integrated once to find $\vec{V}^b$; $\vec{V}^b$ rotated to obtain $\dot{\vec{R}}_m^{\ell}$ and integrated to obtain $\vec{R}_m^{\ell}$.

This system is convenient because $\vec{V}^b$ is the velocity used in aerodynamic calculations. Computation of $\vec{T}^b$ and $\vec{A}^b$ are given in Sections 5 and 6. $\vec{E}^b$ is

$$\vec{E}^b = \left[a_{ij}\right] \vec{E}^{\ell}$$

and $\vec{E}^{\ell}$ is given by (4-9).

4.4 The relative orientation of the b_i axis and ℓ_i axes are shown in Figure 4-1 with an outline of the rocket superimposed.

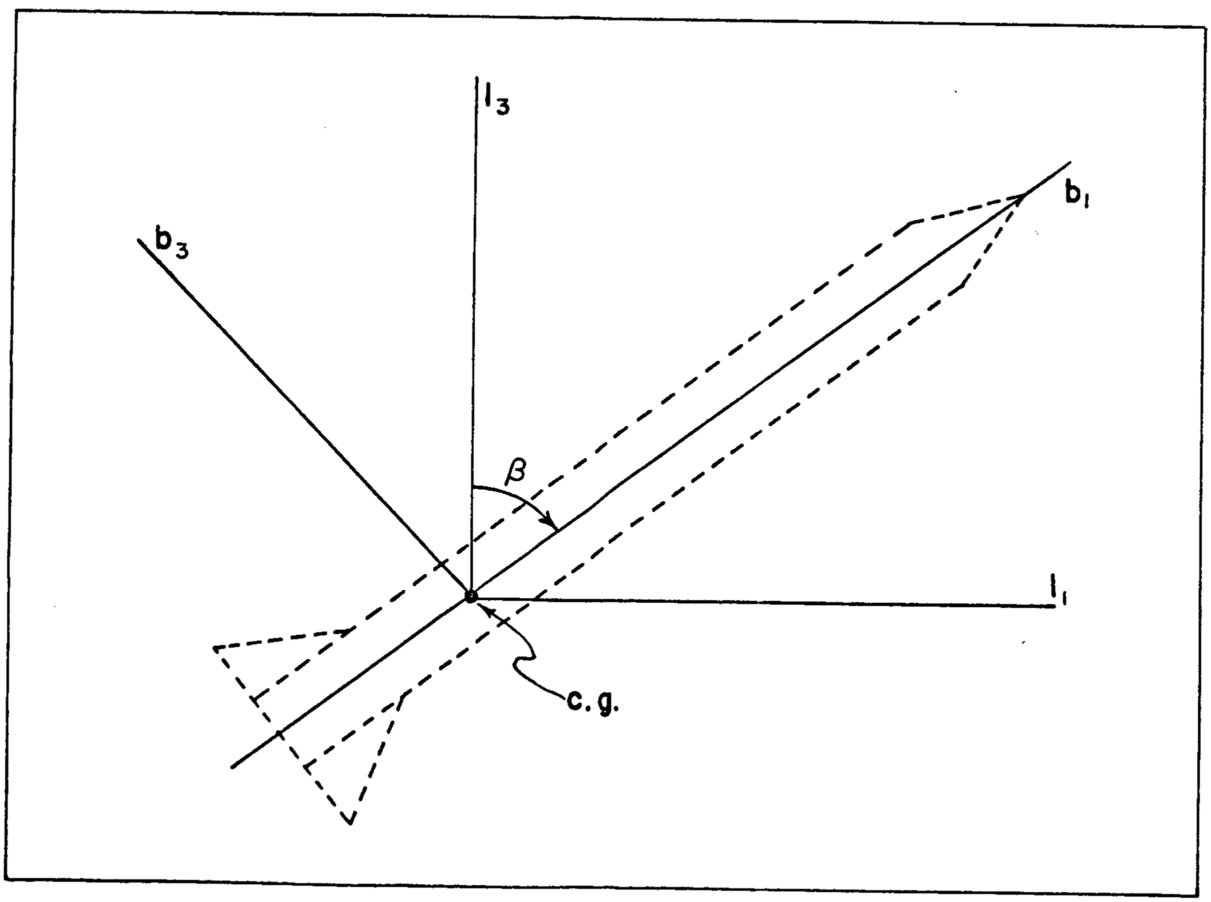


FIGURE 4-1

The b_i system has:

Origin	at the rocket center of gravity (c.g.)
b_1	along the longitudinal axis of the rocket
b_3	normal to b_1 in the pitch plane.

The transformation of a vector between ℓ_i and b_i is

$$\vec{V}^b = \left[R_2\ (\beta - \pi/2)\right] \vec{V}^{\ell}. \qquad 4\text{-}16$$

The rocket is considered to be constrained to move in the b_1 b_3-plane, which is coincident with the ℓ_1 ℓ_3-plane. Force components in the b_2 or ℓ_2 directions are defined to be zero.

4.5 The moment equation for a two-dimensional rigid body reduces to a scalar equation

$$I_2\ \dot{\omega}_2 + \dot{I}_2\ \omega_2 = \Sigma M_2. \qquad 4\text{-}17$$

I_2	Body moment of inertia about the b_2 axis and is discussed in 5.4.
ω_2	Angular velocity of rotation about the b_2 axis and is equal to $\dot{\beta}$.
ΣM_2	Sum of the moments about the b_2 axis.

The moments due to thrust are discussed in Section 5 and those due to aerodynamics in Section 6.

The moment equation is used in the form

$$\ddot{\beta} = \frac{1}{I_2}\left(\Sigma M_2 - \dot{I}_2 \dot{\beta}\right). \qquad 4\text{-}18$$

5. Rocket Functions

5.1 The thrust force magnitude is defined as*

$$|T| = T_s\ (t) + A_E\ (P_S - P_A) \qquad 5\text{-}1$$

where T_S (t) is the sea level thrust of the rocket motor vs. time. The term $A_E\ (P_S - P_A\ (h))$ is a pressure differential correction. A_E is the rocket exit nozzle area. P_A (h) is the ambient pressure, the atmospheric pressure at the rocket altitude h; P_S is sea level standard atmospheric pressure.

The form of the thrust vector equation used in the particle trajectory equations is given in (4-6)

$$\vec{T}^{\ell} = |T|\ \vec{r}_m{}^{\ell}. \qquad 4\text{-}6$$

The thrust vector equations used in the rigid body Programs 2 and 3 is

$$\vec{T}^{b} = |T|\ \vec{b}_1 \qquad 5\text{-}2$$

and in Program 4 the malaligned thrust vector is

$$\vec{T}^{b} = |T|\ (\vec{b}_1 \cos\epsilon\ + \vec{b}_2 \sin\epsilon\ \cos\phi + \vec{b}_3 \sin\epsilon\ \sin\phi).$$

Figure 5-1 shows the geometry of the malaligned thrust. The thrust vector $\vec{T}$ acts, at the center of the rocket motor exit plane, on the b_1 axis a distance R_T from the c.g. The angle between $\vec{T}$ and $\vec{b}_1$ is ϵ, and the plane containing $\vec{T}$

*Reference 3, p. 10.

rotates around $\vec{b}_1$ as the rocket rolls. The roll angle ϕ is defined as the angle between the b_1 b_2-plane and the plane containing $\vec{T}$. Since ϵ is very small and the $\vec{b}_2$ component of $\vec{T}$ is neglected because of the two-dimensional constraint, (5-2) is used in Program 4 in the form

$$\vec{T}^b = |T| \ (\vec{b}_1 + \epsilon \ \sin \phi \ \vec{b}_3) \tag{5-3}$$

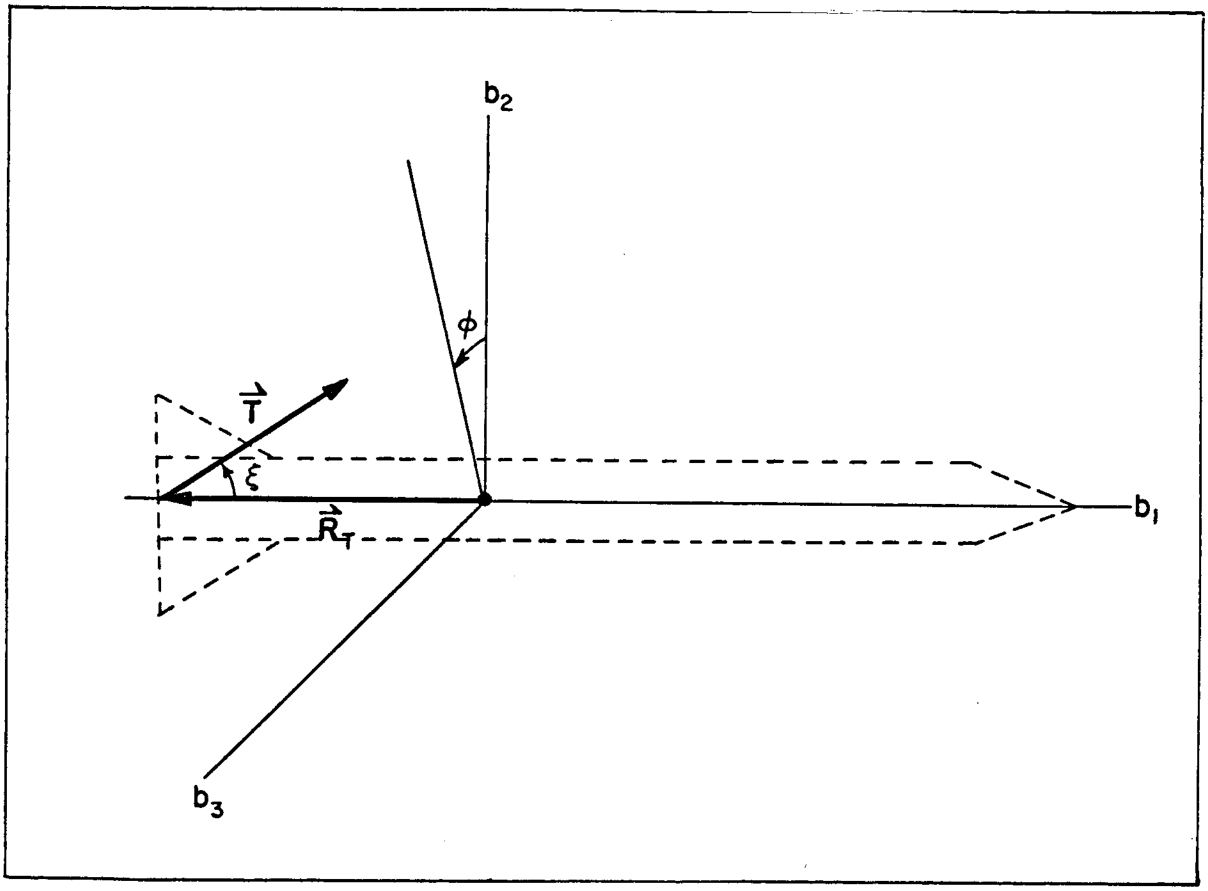


FIGURE 5-1

with ϵ expressed in radians. The moment due to thrust is

$$\vec{M}_T{}^b = \vec{R}_T{}^b \times \vec{T}^b . \tag{5-4}$$

The vector $\vec{R}_T{}^b$ is

$$\vec{R}_T{}^b = -R_T \, \vec{b}_1 . \tag{5-5}$$

Using (5-5) and (5-3), (5-4) reduces to

$$\vec{M}_T{}^b = |T| \epsilon R_T \sin \phi \, \vec{b}_2. \qquad 5\text{-}6$$

5.2 The thrust force and moment defined above is the reaction force for a non-pitching rocket. Another force and the resulting moment, termed jet damping, are caused by the reaction effect if the rocket is pitching. Figure 5-2 illustrates the effect. If the rocket had no pitch angular velocity, the jet gases would have a velocity vector $\vec{V}_E$ with respect to the rocket, with

$$\vec{V}_E = -|V_E| \, \vec{b}_1$$

and the resulting reaction force would be

$$\vec{F}_R = -\dot{M} \, |V_E| \, \vec{b}_1.$$

$\dot{M}$ is the mass rate of flow, taken negative. If the rocket is pitching with an angular velocity $\dot{\beta}$, the exhaust velocity is

$$\vec{V}_{E'} = \vec{V}_E + \dot{\beta} \, R_T \, \vec{b}_3$$

and the reaction force is

$$\vec{F}_{R'} = -\dot{M} \, (|V_E| \, \vec{b}_1 - \dot{\beta} \, R_T \, \vec{b}_3).$$

The component of $\vec{F}_{R'}$ in the $\vec{b}_1$ direction is included in the $\vec{T}$ vector already considered, and the component in the $\vec{b}_3$ direction is the reaction due to jet damping. The moment due to jet damping is

$$\vec{M}_j = \vec{R}_T \times (\dot{M} \dot{\beta} R_T \, \vec{b}_3)$$

$$= \dot{M} \, \dot{\beta} \, (R_T)^2 \, \vec{b}_2. \qquad 5\text{-}7$$

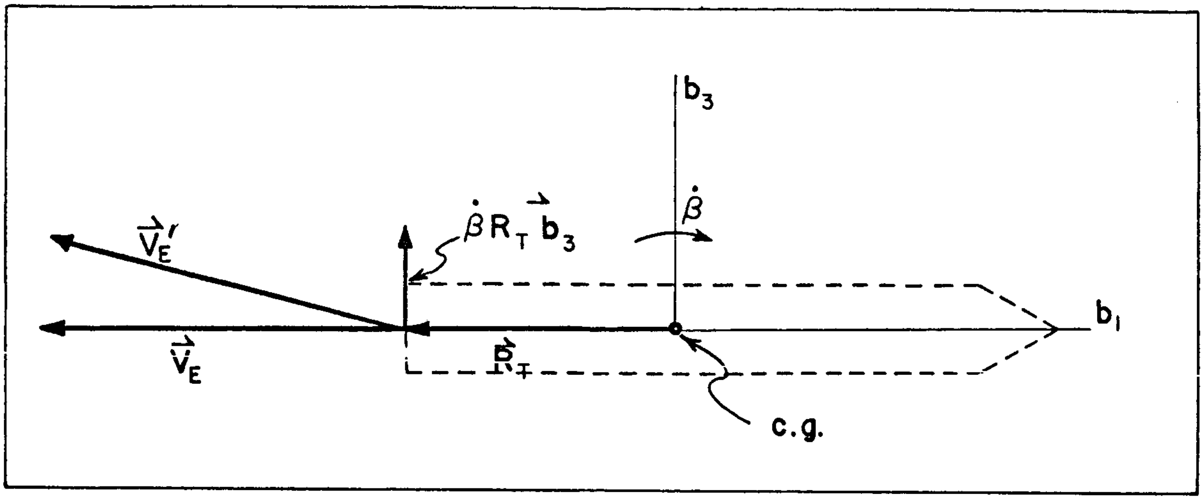


FIGURE 5-2

The last term in equation (4-18) is sometimes included in the jet damping equation*.

5.3 The mass function of the rocket has two types of variation; mass decrease due to motor burning, and mass discontinuities due to stage separation and payload ejection. The mass decrease due to motor burning is set up as follows:

The data available is:

M_I	Mass of rocket at motor ignition t_I
M_B	Mass of rocket at motor burnout t_b
T_s	Sea level thrust vs. time

Assuming that the mass flow rate is proportional to thrust:

$$\dot{M} = k\, T_s$$

$$\int_{t_I}^{t_b} \dot{M}\, dt = -k \int_{t_I}^{t_b} T_s\, dt = -J. \qquad 5\text{-}8$$

*Reference 3, pp. 19-23.

J Total impulse

$$k = -\frac{M_I - M_B}{J}$$

then

$$M(t) = M_I - \int_{t_I}^{t} \dot{M}\, dt$$

$$= M_I + k \int_{t_I}^{t} T_s\, dt. \qquad 5\text{-}9$$

5.4 The center of mass of the rocket is computed from the relationship

$$c.g._{1,2} = \frac{M_1\,(c.g._1) + M_2\,(c.g._2)}{M_1 + M_2} \qquad 5\text{-}10$$

for any two components of the rocket

M_1, M_2 Mass of each component

$c.g._i$ Center of gravity of i^{th} component $(i = 1, 2)$

$c.g._{1,2}$ Center of gravity of the combination.

The moments of inertia are computed by the parallel axis theorem from

$$I_{1,2} = M_1\,(c.g._1 - c.g._{1,2})^2 + M_2\,(c.g._2 - c.g._{1,2})^2 + I_1 + I_2 \qquad 5\text{-}11$$

I_1, I_2 Transverse moment of inertia of each component about its c.g.

$I_{1,2}$ Transverse moment of inertia of the combination about $c.g._{1,2}$.

Data usually furnished for each stage are:

(a) T_s vs. time

(b) Mass with no fuel or payload

(c) Fuel mass

(d) Dimensions

(e) Payload mass.

The mass, c.g. and I data vs. time for each stage are computed using equations (5-9), (5-10) and (5-11), with the empty vehicle as component 1, and the fuel as component 2. The time is referenced to ignition time, and the c.g. location to the motor exit plane.

The stage data are combined, two at a time, using first stage ignition time as reference. The resulting single table of T_s, M, c.g. and I vs. time from first stage ignition is used in the simulation program.

6. Aerodynamic Forces

6.1 Drag and Lift

The Drag and Lift force vectors, $\vec{D}$ and $\vec{L}$, are defined in a relative wind axis system, which in two dimensions is (w_1, w_3). The axis $\vec{w}_1$ is directed along the vector

$$\vec{V}_R = \vec{V} - \vec{W} \qquad 6\text{-}1$$

$\vec{V}_R$	Velocity of the rocket relative to the moving air mass
$\vec{V}$	Velocity of the rocket relative to the earth
$\vec{W}$	Wind velocity relative to the earth.

The axis $\vec{w}_3$ is normal to $\vec{w}_1$ in a right handed sense, and lies in the plane defined by $\vec{w}_1$ and $\vec{b}_1$. The angle between $\vec{b}_1$ and $\vec{w}_1$ is the angle of attack α .

$$\alpha = \cos^{-1} (\vec{b}_1 \cdot \vec{w}_1). \qquad 6\text{-}2$$

In two dimensions, the transformation of any vector $\vec{A}$ from wind axes to body axes is

$$\vec{A}^b = \left[R_2 (\alpha)\right] \vec{A}^w.$$

Figure 6-1 shows the ℓ_i, b_i and w_i axes, with the $\vec{L}$ and $\vec{D}$ vectors. The drag force is

$$\vec{D} = - |D| \vec{w}_1 \qquad 6\text{-}4$$

$$|D| = qd^2 C_D \qquad 6\text{-}5$$

$$q = \gamma /2 \; P_A m^2$$

$$m = V_R/V_O$$

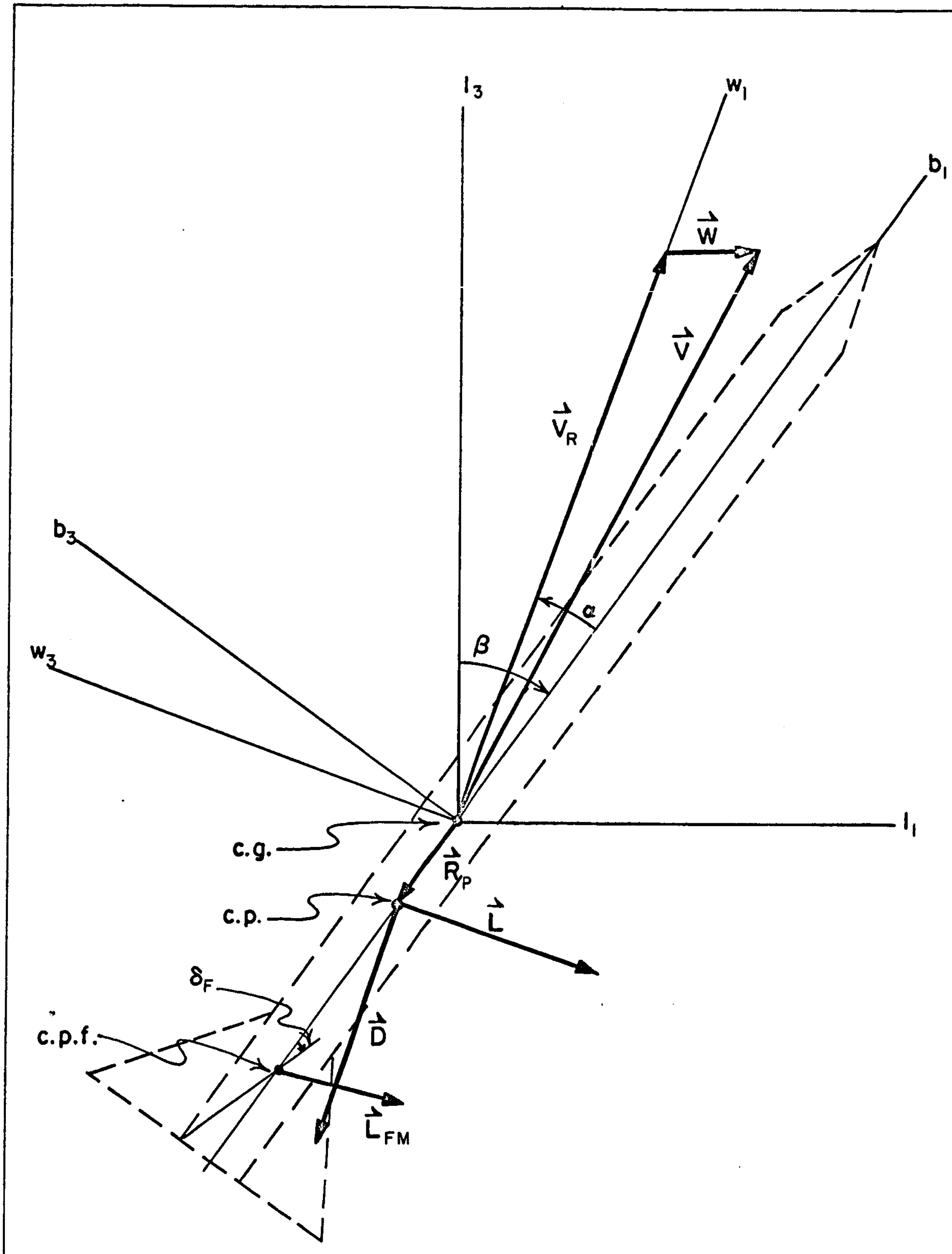
q	Dynamic pressure
γ	Ratio of specific heats for air (1.414)
P_A	Atmospheric pressure as a function of altitude
m	Mach number
V_S	Speed of sound as a function of altitude
d^2	Reference area of the rocket
C_D	Drag coefficient.

The lift force is

$$\vec{L} = - |L| \vec{w}_3 \qquad 6\text{-}6$$

$$|L| = qd^2 C_L. \qquad 6\text{-}7$$

$$C_L = C_{L\alpha} \sin\alpha \cos\alpha \; .$$



l_3
w_1
b_1
$\vec{W}$
$\vec{V}$
$\vec{V}_R$
b_3
w_3
α
β
l_1
c.g.
c.p.
$\vec{R}_P$
$\vec{L}$
δ_F
c.p.f.
$\vec{D}$
$\vec{L}_{FM}$

FIGURE 6-1

C_L Lift coefficient with no fin malalignment

$C_{L\alpha}$ Derivative of the lift coefficient with respect to α evaluated at $\alpha = 0$.

In the programs under discussion C_D and $C_{L\alpha}$ are functions of mach number only, and are obtained from tabular functions

$$C_D = C_D(\mathcal{M}).$$

$$C_L = C_{L\alpha}(\mathcal{M}).$$

$\vec{L}$ and $\vec{D}$ act at the center of pressure location (c.p.)

$$\text{c.p.} = \text{c.p.}(\mathcal{M}).$$

<u>6.2</u> In the particle trajectories α is defined to be zero, the wind axes w_i coincide with b_i and C_L is zero. The aerodynamic force is

$$\vec{A} = -|D|\,\vec{b}_1$$

$$\vec{A} = -qd^2 C_D \dot{\vec{r}}_M{}^{\ell}. \qquad 6\text{-}8$$

In Programs 2 and 3 the lift and drag equation is

$$\vec{A} = \vec{L}^b + \vec{D}^b = -qd^2\left((C_D \cos\alpha - C_{L\alpha} \sin^2\alpha \cos\alpha)\,\vec{b}_1\right.$$

$$\left. + (C_D \sin\alpha + C_{L\alpha} \sin\alpha \cos^2\alpha)\,\vec{b}_3\right). \qquad 6\text{-}9$$

The lift force due to fin malalignment is considered to be independent of the rocket angle of attack and equal to

$$\vec{L}_{FM}{}^{W} = -|L_F|\,(\vec{w}_3 \sin\phi + \vec{w}_2 \cos\phi). \qquad 6\text{-}10$$

ϕ is the rocket roll angle as defined in 5.1.

The rocket is considered to be constrained to move in the b_1, b_3-plane and (6-10) simplifies to

$$\vec{L}_{FM}{}^{w} = -L_F \sin\phi \, \vec{w}_3 \tag{6-11}$$

$$|L_F| = qd^2 \, C_{L\alpha F} \, \delta_F \tag{6-12}$$

$C_{L\alpha F}$ The derivative with respect to α of the lift coefficient of the fins alone

δ_F The angle of malalignment of the fins.

In Program 4 the lift and drag equation is

$$\vec{A} = \vec{L}^b + \vec{D}^b + \vec{L}_{FM}{}^b$$

$$= -qd^2 \left((C_D \cos\alpha - (C_{L\alpha} \sin\alpha \cos\alpha + C_{L\alpha F} \, \delta_F \sin\phi) \sin\alpha) \vec{b}_1 + (C_D \sin\alpha + (C_{L\alpha} \sin\alpha \cos\alpha + C_{L\alpha F} \, \delta_F \sin\phi) \cos\alpha) \vec{b}_3 \right). \tag{6-13}$$

<u>6.3</u> The moment due to lift and drag in Programs 2 and 3

$$\vec{M}_{L,D}{}^b = \vec{R}_P{}^b \times (\vec{L}^b + \vec{D}^b) \tag{6-14}$$

$$\vec{R}_P{}^b = -|R_P| \, \vec{b}_1$$

$$R_P = c.g. - c.p.$$

c.g.(t) Center of gravity distance from motor exit plane

c.p.(m) Static center of pressure distance from exit plane.

In Program 4, the additional moment due to fin malalignment is

$$\vec{M}_{FM}{}^{b} = \vec{R}_{PF}{}^{b} \times \vec{L}_{FM}{}^{b} \qquad \text{6-15}$$

$$\vec{R}_{PF}{}^{b} = -\left|R_{PF}\right| \vec{b}_1$$

$$R_{PF} = \text{c.g.} - \text{c.p.f.}$$

c.p.f. Center of pressure of fins.

6.4 In addition to the lift and drag forces and moments, which are functions of angle of attack, a pitching rocket experiences a pitch damping moment. Figure 6-2 illustrates the situation for a rocket with relative velocity $\vec{V}_R$, angle of attack α and pitch rate $\dot{\beta}$. The center of pressure of the tail is at $-x_T$ from the c.g. and is moving in the b_3 direction with a velocity $x_T\dot{\beta}$, so that the tail "sees" an angle of attack which is approximated by

$$\alpha_T = \alpha + \frac{x_T\dot{\beta}}{V_R} .$$

The lift force on the tail is

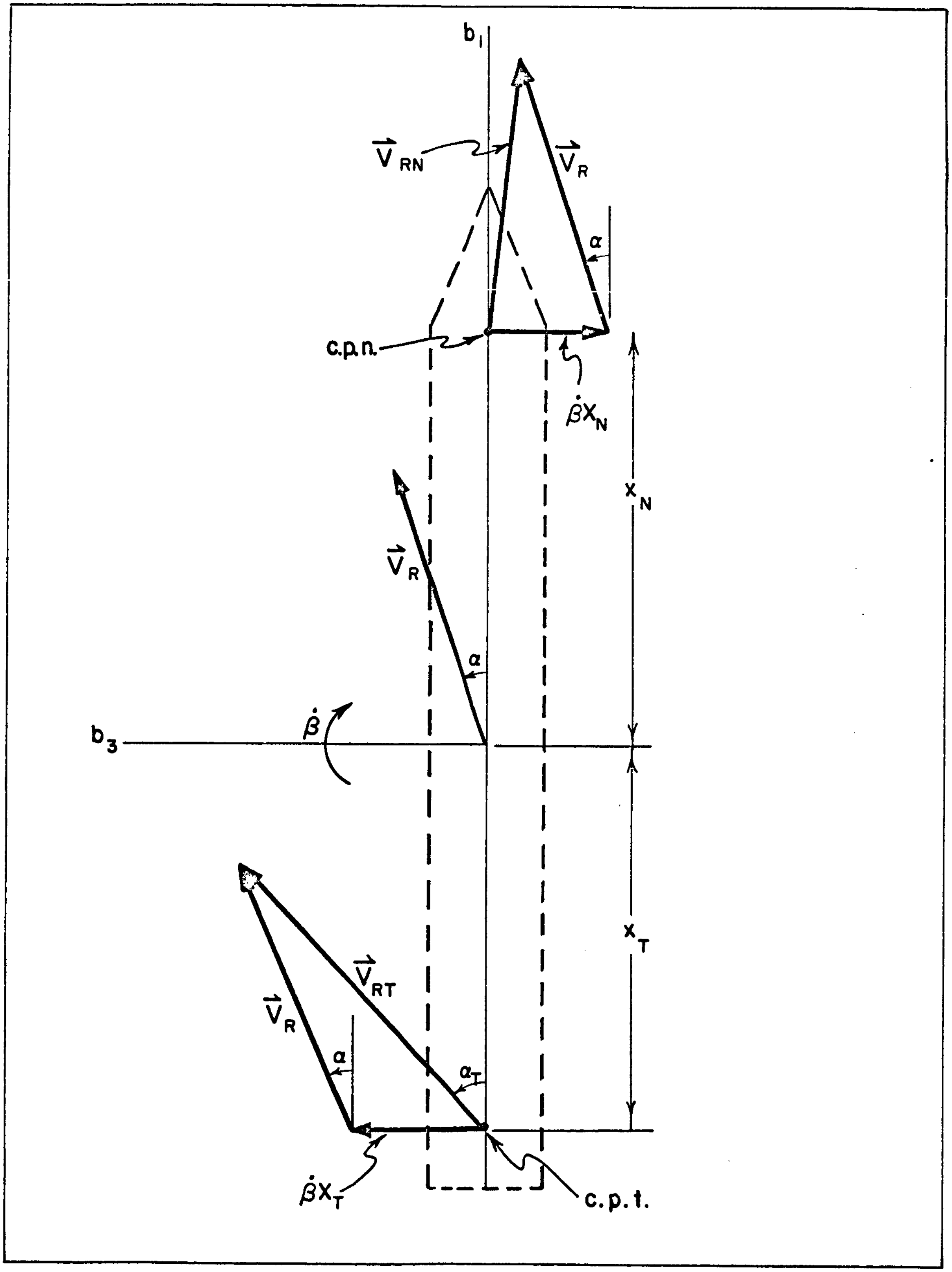
$$\vec{L}_T{}^{\vec{b}} = \vec{L}_T{}^{\vec{b}}(\alpha) - qd^2 C_{L\alpha T} \frac{x_T\dot{\beta}}{V_R} \vec{b}_3.$$

The second term on the right is a lift due to pitch rate. The tail surface contribution to moment due to pitch rate is

$$\vec{M}_{\dot{\beta} T} = -qd^2 C_{L\alpha T} x_T{}^2 \frac{\dot{\beta}}{V_R} \vec{b}_2.$$

Similarly, the nose contribution to moment due to pitch rate is

$$\vec{M}_{\dot{\beta} T} = -qd^2 C_{L\alpha N} x_N{}^2 \frac{\dot{\beta}}{V_R} \vec{b}_2.$$



b_1
$\vec{V}_{RN}$
$\vec{V}_R$
α
c.p.n.
$\dot{\beta}X_N$
x_N
$\vec{V}_R$
α
$\dot{\beta}$
b_3
x_T
$\vec{V}_{RT}$
$\vec{V}_R$
α
α_T
$\dot{\beta}X_T$
c.p.t.

FIGURE 6-2

The contributions of all aerodynamic elements can be summed to produce

$$\vec{M}_{\dot{\beta}} = -qd^2 \left[\sum_i C_{L\alpha\, i}\, x_i{}^2\right] \frac{\dot{\beta}}{V_R}\, \vec{b}_2. \qquad 6\text{-}16$$

In Programs 2, 3, and 4, $\vec{M}_{\dot{\beta}}$ is computed by

$$\vec{M}_{\dot{\beta}} = -qd^2\, x_T{}^2\, C_{L\alpha T} \frac{\dot{\beta}}{V_R}\, \vec{b}_2. \qquad 6\text{-}17$$

The assumption that pitch damping moment, $\vec{M}_{\dot{\beta}}$, is entirely caused by the tail surfaces is justified for fin stabilized rockets with large stability margins.

7. Earth Forces

The set of equations used in Program 5 describe very accurately the gravitational force of the earth and the centrifugal and Coriolis forces caused by the rotation of the earth. The force equations are derived in this section. The coordinate transformations used are described here, and derived in Section 8.

7.1 The effective force exerted by the earth on a body of mass M, as observed in a coordinate system g_i fixed with respect to the rotating earth, is

$$M\vec{E}^{g} = M\left\{ \vec{G}^{g} - 2\,(\vec{\omega}^{g} \times \dot{\vec{\rho}}^{g}) - (\vec{\omega}^{g} \times (\vec{\omega}^{g} \times \vec{\rho}^{g})) \right\} \qquad \text{7-1}$$

M	Mass of the body
$\vec{G}$	Gravitational acceleration
$\vec{\omega}$	Angular velocity of the earth
$\dot{\vec{\rho}}^{g}$	Velocity of the body in the coordinate system fixed with respect to the earth
$\vec{\rho}^{g}$	Position vector of the body with respect to the center of the earth

$$|\omega| = 7.292115851 \quad 10^{-5} \text{ rad. sec.}^{-1}$$

The term $-2\,(\vec{\omega}^{g} \times \dot{\vec{\rho}}^{g})$ is the Coriolis acceleration $\mathcal{C}$, and $-\vec{\omega}^{g} \times (\vec{\omega}^{g} \times \vec{\rho}^{g})$ is the centrifugal acceleration C.

7.2 The earth's gravitational potential at a point with geocentric coordinates (ρ , $\mathscr{L}$, Λ) as defined in (8-2) is given in Reference 4 as

$$\Phi = \frac{-k_c^2 m_1 a}{\rho} \left\{ 1 + \frac{J_2}{2} \frac{a^2}{\rho^2} (1 - 3 \sin^2 \mathscr{L}) \right.$$

$$+ \frac{J_3}{2} \frac{a^2}{\rho^3} (3 - 5 \sin^2 \mathscr{L}) \sin \mathscr{L}.$$

$$\frac{-J_4 a^4}{8 \rho^4} (3 - 30 \sin^2 \mathscr{L} + 35 \sin^4 \mathscr{L})$$

$$\frac{-J_5 a^5}{8 \rho^5} (15 - 70 \sin^2 \mathscr{L} + 63 \sin^4 \mathscr{L}) \sin \mathscr{L}$$

$$\left. \frac{+J_6 a^6}{16 \rho^6} (5 - 105 \sin^2 \mathscr{L} + 315 \sin^4 \mathscr{L} - 231 \sin^6 \mathscr{L}) \right\}. \qquad 7\text{-}2$$

The terms with coefficients J_2 to J_6 are the zonal harmonics and give the effect of the equatorial bulge. The terms with coefficients J_3 and J_5 take into account the asymmetry with respect to the equator as computed from satellite orbit perturbations. The potential is symmetric with respect to the polar axis, so no terms in Λ appear. Local irregularities, such as continents, mountains, and oceans are not considered, and the gravitational force computed from (7-2) for any specific point will differ slightly from the force measured at the point.

Rewriting equation (7-2) yields:

$$\Phi = \frac{k_{G1} a^2}{\rho} \left[1 + k_{G2} \left(\frac{a}{\rho}\right)^2 (1 - 3 \sin^2 \mathscr{L}) \right.$$

$$+ k_{G3} \left(\frac{a}{\rho}\right)^3 (3 - 5 \sin^2 \mathscr{L}) \sin \mathscr{L}$$

$$- k_{G4} \left(\frac{a}{\rho}\right)^4 (3 - 30 \sin^2 \mathscr{L} + 35 \sin^4 \mathscr{L})$$

$$- k_{G5} \left(\frac{a}{\rho}\right)^5 (15 - 70 \sin^2 \mathscr{L} + 63 \sin^4 \mathscr{L}) \sin \mathscr{L}$$

$$\left. + k_{G6} \left(\frac{a}{\rho}\right)^6 (5 - 105 \sin^2 \mathscr{L} + 315 \sin^4 \mathscr{L} - 231 \sin^6 \mathscr{L}) \right] \qquad 7\text{-}3$$

$k_{G_1} = -32.146484$ ft/sec.^{2}

$a = 20{,}925{,}645$ feet

$k_{G_2} = .00054114$

$k_{G_3} = -.00000115$

$k_{G_4} = -.000000265$

$k_{G_5} = -.000000025$

$k_{G_6} = .0000000625$

<u>7.3</u> The gravitational acceleration caused by Φ is

$$G = -\Delta \Phi \ . \tag{7-4}$$

The vector $\vec{G}$ (M), the gravitational acceleration at a point M with geocentric coordinates ρ_M, L_M, Λ_M is evaluated in a Cartesian coordinate system m_i with:

Origin at M

$\vec{m}_3$ directed along the radius vector from the center of the earth to M

$\vec{m}_2$ in the meridian plane of M.

Evaluating the gradient operator of equation (7-4) at the origin of the m_i system yields

$$\vec{G}^m = -\frac{1}{\rho_M \cos \mathcal{L}_L} \frac{\partial \Phi}{\partial \Lambda}\bigg|_M \vec{m}_1 - \frac{1}{\rho_M} \frac{\partial \Phi}{\partial \mathcal{L}}\bigg|_M \vec{m}_2 - \frac{\partial \Phi}{\partial \rho}\bigg|_M \vec{m}_3 \, . \tag{7-5}$$

In the notation defined in 9.1.2,

$$\vec{G}^m = G_1{}^m \, \vec{m}_1 + G_2{}^m \, \vec{m}_2 + G_3{}^m \, \vec{m}_3.$$

From equation 7-5, the components of $\vec{G}^m$ at the point M are

$$G_1{}^m = \frac{-1}{\rho \cos \mathscr{L}} \frac{\partial \bar{\Phi}}{\partial \Lambda} = 0$$

$$G_2{}^m = \frac{-1}{\rho} \frac{\partial \Phi}{\partial \mathscr{L}}$$

$$G_3{}^m = - \frac{\partial \Phi}{\partial \rho}.$$

Taking the indicated partial derivatives of Φ from (7-3) yields:

$$G_2{}^m = k_{G1} \left(\frac{a}{\rho_M}\right)^2 \Big[+k_{G2} \left(\frac{a}{\rho_M}\right)^2 (6 \sin \mathscr{L}_M \cos \mathscr{L}_M)$$

$$+ k_{G3} \left(\frac{a}{\rho_M}\right)^3 (3 - 15 \sin^2 \mathscr{L}_M) \cos \mathscr{L}_M$$

$$+ k_{G4} \left(\frac{a}{\rho_M}\right)^4 (60 \sin \mathscr{L}_M - 140 \sin^3 \mathscr{L}_M) \cos \mathscr{L}_M$$

$$- k_{G5} \left(\frac{a}{\rho_M}\right)^5 (15 - 210 \sin^2 \mathscr{L}_M + 315 \sin^4 \mathscr{L}_M) \cos \mathscr{L}_M$$

$$+ k_{G6} \left(\frac{a}{\rho_M}\right)^6 (210 \sin \mathscr{L}_M - 1260 \sin^3 \mathscr{L}_M + 1386 \sin^5 \mathscr{L}_M) \cos \mathscr{L}_M \Big] \quad \text{7-6}$$

$$G_3^m = k_{G1}\left(\frac{a}{\rho_M}\right)^2 \Big[1 + k_{G2}\left(\frac{a}{\rho_M}\right)^2 (3 - 9\sin^2\mathscr{L}_M)$$

$$- k_{G3}\left(\frac{a}{\rho_M}\right)^3 (12 - 20\sin^2\mathscr{L}_M)\sin\mathscr{L}_M$$

$$+ k_{G4}\left(\frac{a}{\rho_M}\right)^4 (15 - 150\sin^2\mathscr{L}_M + 175\sin^4\mathscr{L}_M)$$

$$+ k_{G5}\left(\frac{a}{\rho_M}\right)^5 (90 - 420\sin^2\mathscr{L}_M + 378\sin^4\mathscr{L}_M)\sin\mathscr{L}_M$$

$$+ k_{G6}\left(\frac{a}{\rho_M}\right)^6 (37 - 735\sin^2\mathscr{L}_M + 2205\sin^4\mathscr{L}_M - 1617\sin^6\mathscr{L}_M)\Big]. \qquad 7\text{-}7$$

7.4 The centrifugal acceleration $\vec{C}$ in the m_i system is

$$\vec{C}^m = -\vec{\omega}^m \times \vec{\omega}^m \times \vec{\rho}_M^m \qquad 7\text{-}8$$

$$\vec{\omega}^m = |\omega|\cos\mathscr{L}_M\,\vec{m}_2 + |\omega|\sin\mathscr{L}_M\,\vec{m}_3 \qquad 7\text{-}9$$

$$\vec{\rho}_M^m = \rho_M\,\vec{m}_3. \qquad 7\text{-}10$$

ρ_M Distance from the center of the earth to M.

In component form

$$\vec{C}^m = C_1^m\,\vec{m}_1 + C_2^m\,\vec{m}_2 + C_3^m\,\vec{m}_3$$

using (7-9) and (7-10) in (7-8) yields the components:

$$C_1^m = 0$$

$$C_2^m = -\omega^2\rho_M\sin\mathscr{L}_M\cos\mathscr{L}_M \qquad 7\text{-}11$$

$$C_3^m = \omega^2 \rho_M \cos^2 \mathcal{L}_M . \quad 7\text{-}12$$

7.5 Under the combined gravitational and centrifugal forces, the earth has assumed a shape normal to the resultant of $\vec{G} + \vec{C}$. This shape is very nearly an ellipsoid of revolution. The internationally adopted dimensions are*

Semi-major axes	$a = b = 20{,}925{,}647$ feet
Semi-minor axis	$c = 20{,}855{,}497$ feet
Flattening	$\dfrac{a - c}{a} = \dfrac{1}{298.3}$.

The equation for any point M, with centric coordinates $\rho_{M'}$, $\mathcal{L}_{M'}$ on the surface of this ellipsoid is

$$\rho_{M'} = \frac{a}{\sqrt{1+k \sin^2 \mathcal{L}_{M'}}}$$

where

$$k = \frac{a^2 - c^2}{c^2} = .00673852.$$

7.6 The combined vector $\vec{G}^m + \vec{C}^m$ is the plumb line direction $\vec{H}^m$, and is transformed into the ℓ_1 system by the transformation derived in 8.7. Then the Coriolis acceleration $\vec{\mathcal{C}}$ is computed by

$$\vec{\mathcal{C}}^{\ell} = -2\left[\vec{\omega}^{\ell} \times \dot{R}_m^{\ell}\right]$$

where

$$\vec{\omega}^{\ell} = |\omega| \left[\cos L_L \vec{\ell}_2 + \sin L_L \vec{\ell}_3\right].$$

The earth force vector in the ℓ_i system is

$$M\vec{E}^{\ell} = M\left[\vec{\mathcal{C}}^{\ell} + \vec{H}^{\ell}\right].$$

*Reference 4.

The earth force vector in the ℓ_i system is

$$M\vec{E}^{\ell} = M\left[\vec{\mathcal{C}}^{\ell} + \vec{H}^{\ell}\right] \tag{7-13}$$

and in the f_i system (paragraph 8.9) is

$$M\vec{E}^{f} = M\left[R_2\left(\alpha_f - 90°\right)\right]\vec{E}^{\ell}. \tag{7-14}$$

8. Coordinate Transformation on the Earth's Surface

8.1 Trajectory data are computed by Program 5 in a Cartesian system ℓ_i with origin at the launcher and with the $\ell_1 \ell_2$-plane normal to the direction of a plumb bob at the origin. Three types of coordinate transformations which are used in the computation of data in Program 5 are listed below.

8.1.1 The atmospheric data used in the program are tabulated as functions of height above the sea level surface of the curved earth. In order to use these data it is necessary to compute the height of the rocket, given the rocket coordinates in the ℓ_i system. In addition, the data on a long range trajectory are often more meaningful when the rocket position is given in terms of latitude and longitude on the earth's surface. These three coordinates--latitude, longitude and height--are termed geodetic coordinates. Geodetic coordinates are described in terms of the geometry of the earth in Section 8.3. The transformation from a Cartesian system such as ℓ_i is accomplished by using geocentric coordinates as an intermediate step. Geocentric coordinates are geometrically described in Section 8.2. The transformation of data from the ℓ_i system to geocentric and then to geodetic coordinates is derived in Section 8.6.

8.1.2 The equations which describe the gravitational force field of the earth and the centrifugal and Coriolis forces due to the rotation of the earth are easily derived in a Cartesian coordinate system m_i with:

Origin	at the rocket position
m_3	axis in the direction of the geocentric radius vector
m_2 m_3-plane	co-planar with the meridian plane of the rocket.

The transformation from the ℓ_i system to the m_i system is given in 8.7.

8.1.3 It is sometimes required to give the rocket trajectory data in a Cartesian system t_i, with origin not located at the launcher. In general, this coordinate system will have the t_1 t_2-plane normal to the plumb line direction at the origin. The transformation from the ℓ_1 system to the t_1 system is given in 8.8.

8.2 Geocentric spherical coordinates are shown in Figure 8-1. The origin at the earth's center is point C; the polar axis is CN; the reference (Greenwich) meridian is the arc AN; the local meridian is the arc BN. The geocentric spherical coordinates of a point M are ρ_M, Λ_M, $\mathscr{L}_M$. ρ_M is the magnitude of the radius vector from the earth's center to M. $\mathscr{L}_M$ is the geocentric latitude of M, and is the angle between the vector ρ_M and CB, the projection of ρ_M on the equatorial plane, measured positive north from the equator. Λ_M is the geocentric longitude of M, and is the dihedral angle between the local meridian plane and the reference meridian plane, measured positive west. The solid arcs on Figure 8-1 indicate a geocentric sphere with radius the semi-major axis of the earth, a, and the dotted lines indicate the relationship of the earth's surface to such a sphere. The earth's surface is considered to be an ellipsoid of revolution with semi-major axis, a, and semi-minor axis c. Section 8.4 gives the dimensions adopted.

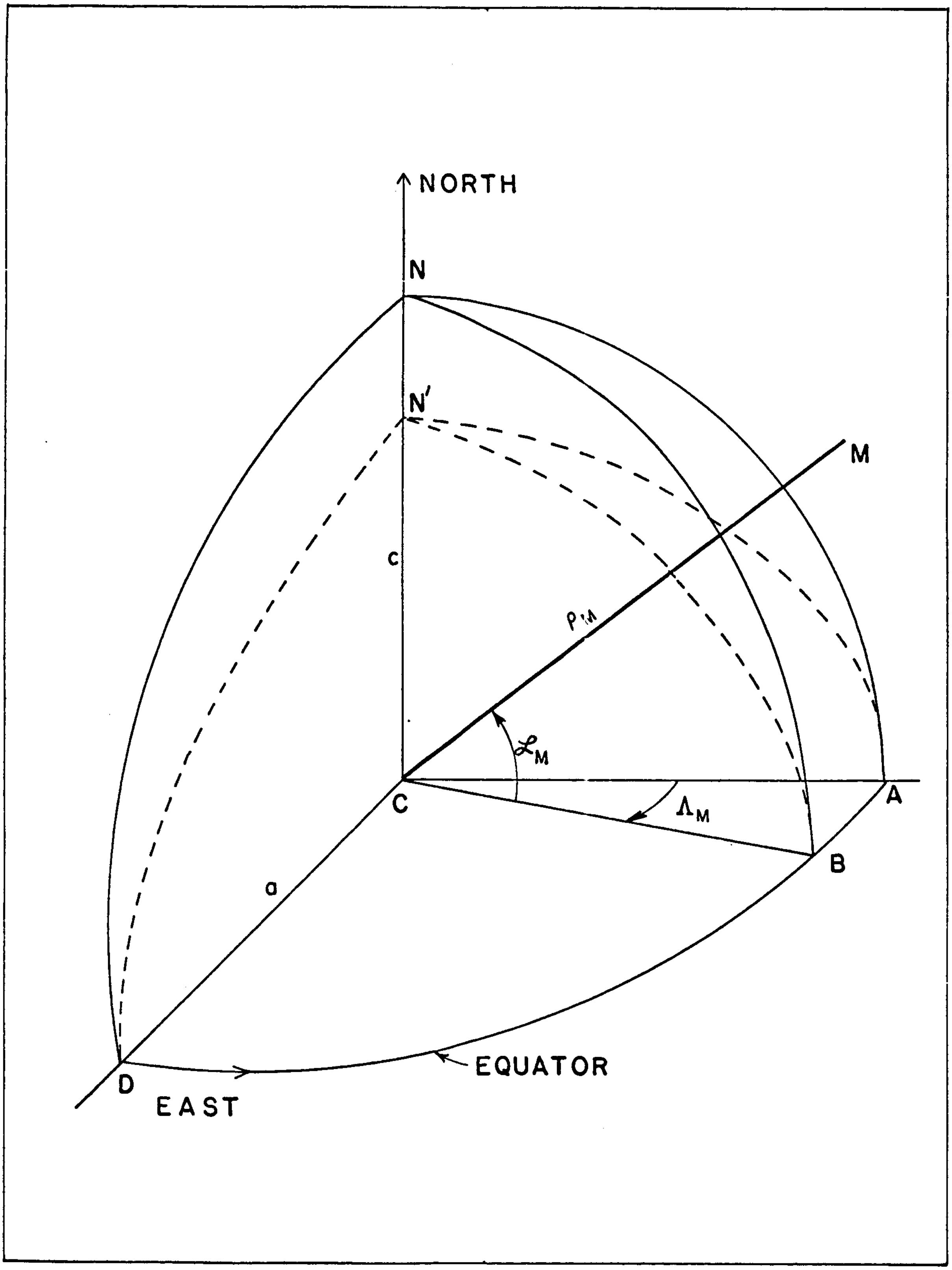


FIGURE 8-1

8.3 Geodetic coordinates are coordinates which are measured with respect to the earth's surface. Geodetic height is distance from the sea level surface in the plumb line direction. The plumb line direction at any point is the direction of the resultant of the gravitational and centrifugal forces at that point. Strictly speaking, the sea level surface is the geometrical shape which at any point is normal to the plumb line direction at that point and coincides with the average position of the surface of the oceans. The sea level surface is approximated by the ellipsoid of revolution defined by equation (8-1). Geodetic coordinates are shown in Figure 8-2. The surface ADN' is the octant of the ellipsoid. C is the center of the earth, N' the north pole; the arc AN' is the reference meridian and the arc BN' the local meridian. The point M' is the foot of the perpendicular from M to the ellipsoid. The geodetic coordinates of M are H_M, L_M, Λ_M. The geodetic height H_M is the length of the line, M'M; the geodetic latitude, L_M is the angle between the equatorial plane and the extension of M'M; and Λ_M, the longitude of M is the same in geocentric and geodetic coordinates.

8.4 The earth's surface is defined* to be an ellipsoid of revolution with

$$a = 20{,}925{,}647 \text{ feet}$$

$$c = 20{,}855{,}497 \text{ feet}$$

$$\frac{a - c}{a} = \frac{1}{298.3}.$$

The equation for this ellipse can be written

$$\rho = \frac{a}{\sqrt{1+k \sin^2 \mathcal{L}}} \qquad 8\text{-}1$$

*Reference 4.

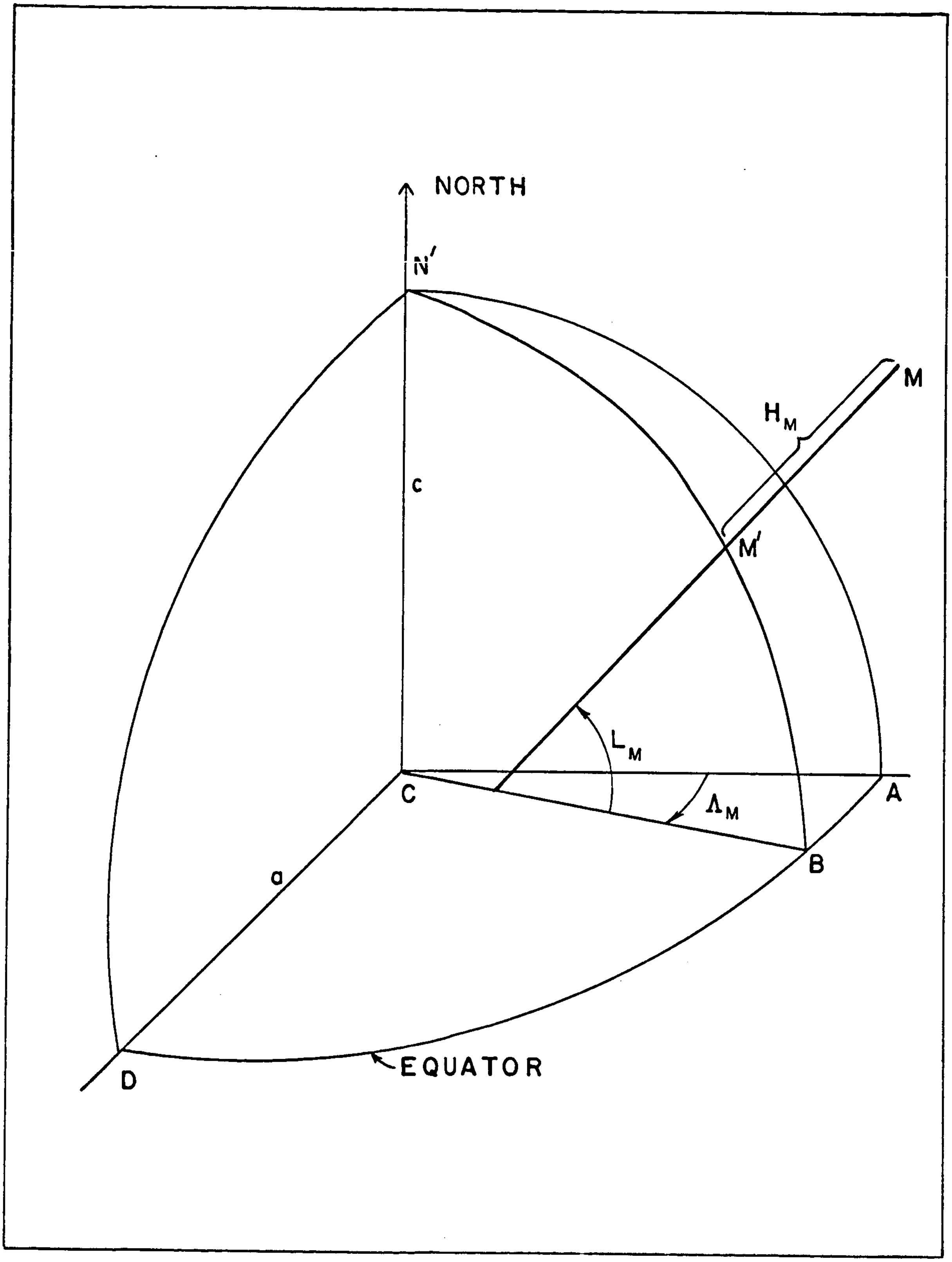


FIGURE 8-2

where

$$k = \frac{a^2 - c^2}{c^2} = .00673852. \qquad 8\text{-}2$$

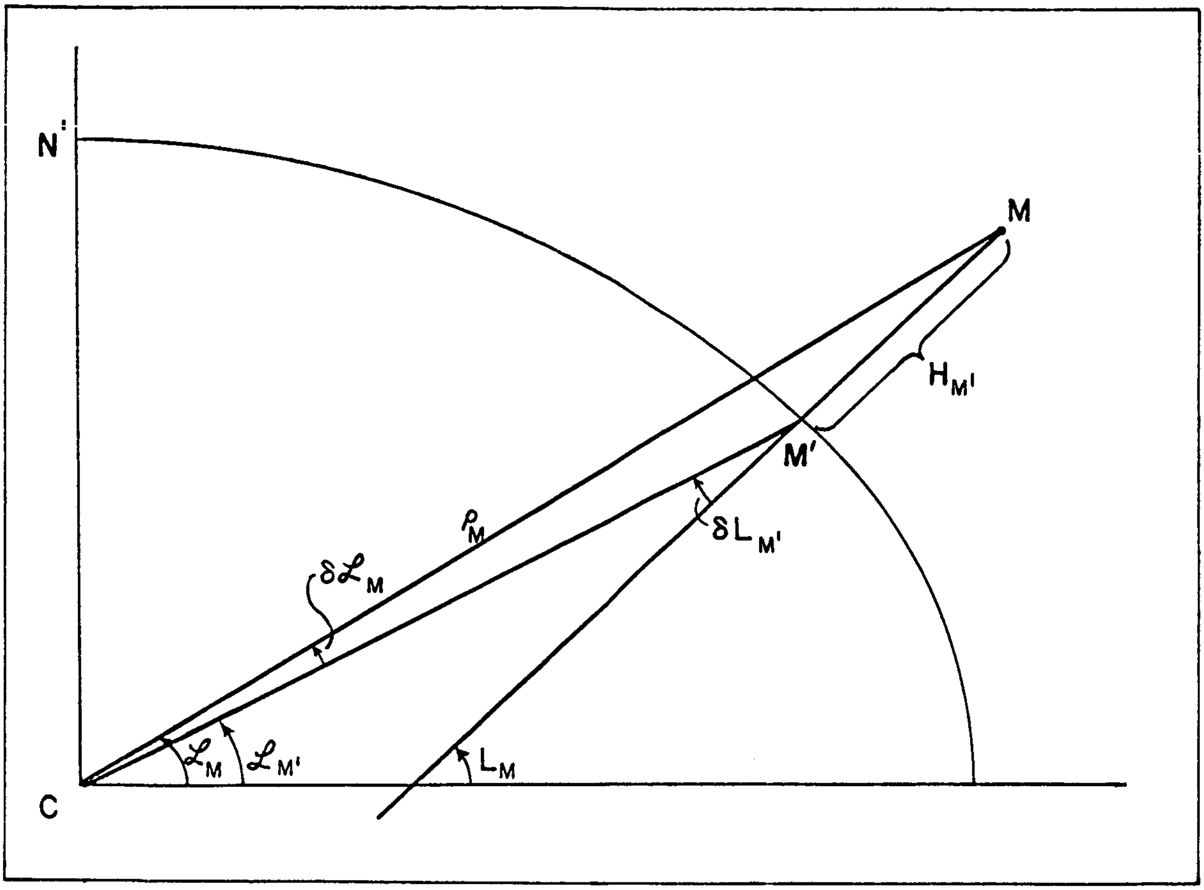


FIGURE 8-3

8.4.1 Figure 8-3 shows both geodetic and geocentric coordinates of points M and M' in the local meridian plane. The difference between geocentric and geodetic latitudes of M' is $\delta L_{M'}$

$$\delta L_{M'} = L_M - \mathscr{L}_{M'}.$$

The difference between the geocentric latitudes of M and of M' is $\delta \mathscr{L}_M$

$$\delta \mathscr{L}_M = \mathscr{L}_M - \mathscr{L}_{M'}.$$

A general equation for $\delta \mathcal{L}$ in terms of $\mathcal{L}$ and L is used to show the relation between geodetic and geocentric coordinates.

Figure 8-4 illustrates the geometry by which the equation for δL_P can be derived. P is any point on the surface of the ellipsoid and the arc AA' is a segment of a geocentric circle with radius ρ $(\mathcal{L}_P)$. δL_P is the angle between SS' and AA' at P. From the figure

$$\tan \delta L_P = \lim_{\Delta \to 0} \frac{-\Delta \rho}{\rho_P \Delta \mathcal{L}}$$

$$= \frac{1}{\rho_P} \lim_{\Delta \mathcal{L} \to 0} \frac{-\Delta \rho}{\Delta \mathcal{L}}$$

$$= \frac{1}{\rho_P} \left(- \frac{d\rho}{d\mathcal{L}} \right) \Bigg|_{\rho = \rho_P} .$$

Taking $d\rho / d\mathcal{L}$ in equation (8-1) and dividing by ρ yields

$$\tan \delta L_P = \frac{k \sin \mathcal{L}_P \cos \mathcal{L}_P}{1+k \sin^2 \mathcal{L}_P} \qquad 8\text{-}4$$

or

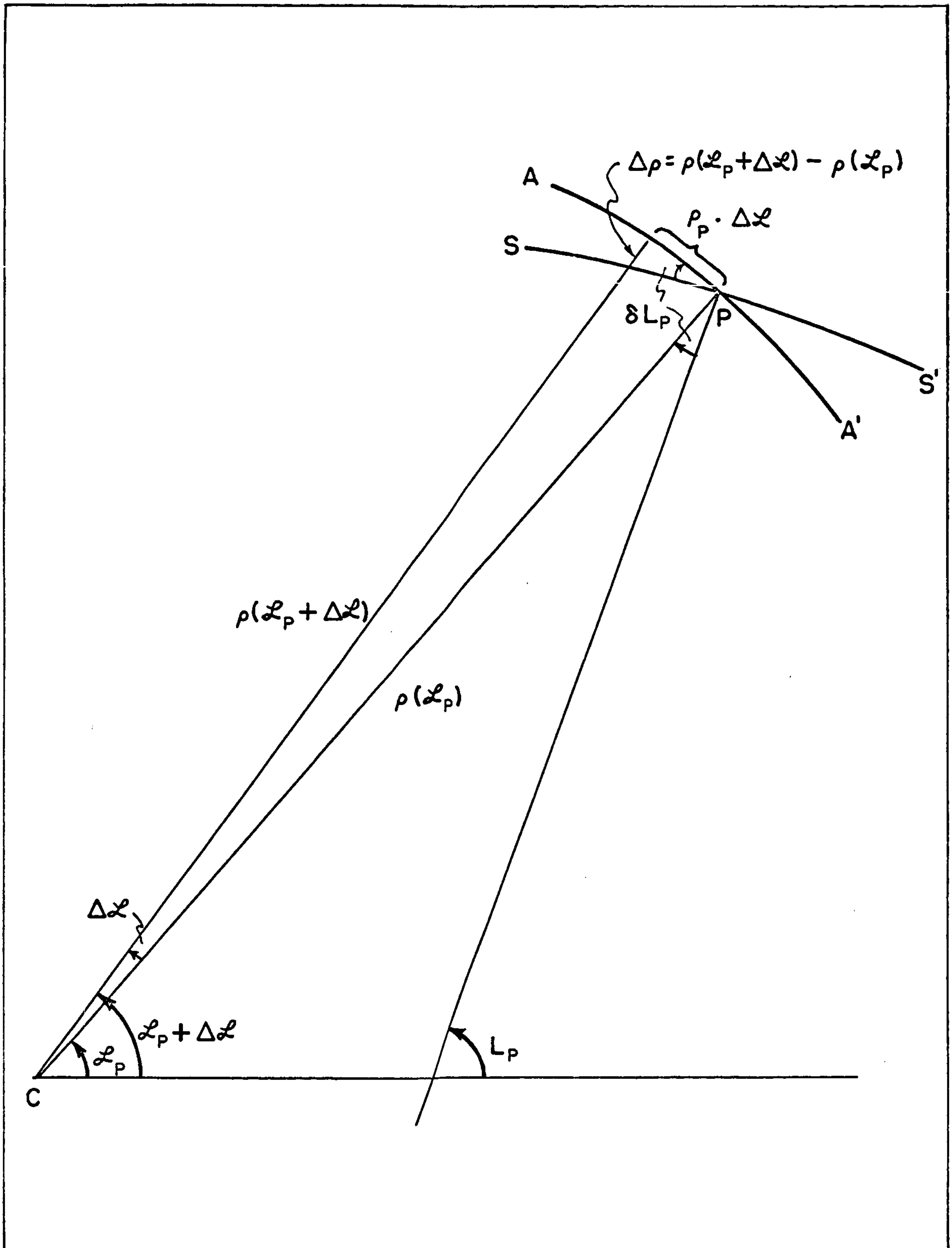
$$\tan \delta L = \frac{k \sin \mathcal{L} \cos \mathcal{L}}{1+k \sin^2 \mathcal{L}}$$

for any point on the surface. To find δ L as a function of L let $\mathcal{L} = L - \delta L$ in (8-4)

$$\tan \delta L = \frac{k \sin (L - \delta L) \cos (L - \delta L)}{1+k \sin^2 (L - \delta L)}$$

and by trigonometric identities

$$\frac{\sin 2 \delta L}{1 + \cos 2 \delta L} = \frac{\frac{k}{2} \sin (2L - 2 \delta L)}{1 + \frac{k}{2} (1 - \cos (2L - 2 \delta L))} .$$



$\Delta\rho = \rho(\mathcal{L}_P + \Delta\mathcal{L}) - \rho(\mathcal{L}_P)$
A
$\rho_P \cdot \Delta\mathcal{L}$
S
δL_P
P
S'
A'
$\rho(\mathcal{L}_P + \Delta\mathcal{L})$
$\rho(\mathcal{L}_P)$
$\Delta\mathcal{L}$
$\mathcal{L}_P + \Delta\mathcal{L}$
$\mathcal{L}_P$
L_P
C

FIGURE 8-4

Expanding the right side in terms of sine and cosine of 2L and $2\delta L$, clearing fractions and collecting terms in $2\delta L$ yields

$$\tan \delta L = \frac{k \sin L_M \cos L_M}{1+k \cos^2 L_M} . \tag{8-5}$$

8.5 Given geodetic coordinates of a point M, (H_M, L_M, Λ_M), to find the geocentric coordinates $(\rho_M, \mathscr{L}_M, \Lambda_M)$, compute

$$\delta L_{M'} = \tan^{-1} \frac{k \sin L_M \cos L_M}{1+k \cos^2 L_M}$$

$$\mathscr{L}_{M'} = L_M - \delta L_{M'} \tag{8-6}$$

$$\rho_{M'} = \frac{a}{\sqrt{1+k \sin^2 \mathscr{L}_{M'}}} . \tag{8-1}$$

From the triangle MCM' in Figure 8-3

$$\rho_M = \sqrt{\rho_{M'}^2 + H_M^2 + 2\rho_{M'} H_M \cos \delta L_M} \tag{8-7}$$

or

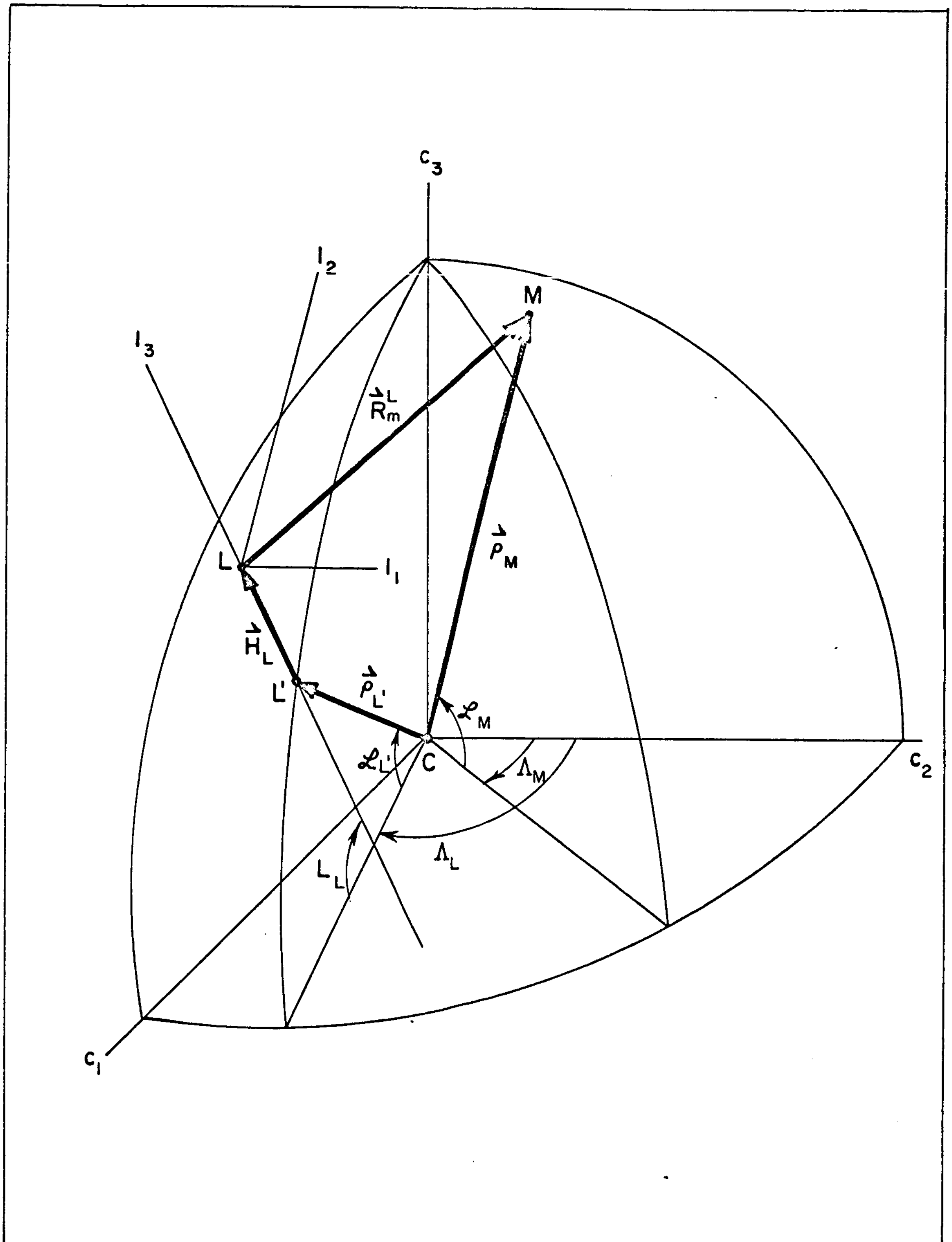
$$\rho_M = \sqrt{(\rho_{M'} + H_M \cos \delta L_{M'})^2 + (H_M \sin \delta L_{M'})^2}$$

$$\delta \mathscr{L}_M = \sin^{-1} \left(\frac{H_M \sin \delta L_{M'}}{R_M} \right) \tag{8-8}$$

$$\mathscr{L}_M = \mathscr{L}'_M + \delta \mathscr{L}_M . \tag{8-9}$$

8.6 The problem of transformation stated in 8.1.1 above is restated below:

Given a Cartesian coordinate system ℓ_1, ℓ_2, ℓ_3 defined as follows:



c_3
l_2
l_3
M
$\vec{R}_m^L$
$\vec{\rho}_M$
L
l_1
$\vec{H}_L$
L'
$\vec{\rho}_{L'}$
$\mathcal{L}_M$
$\mathcal{L}_{L'}$
C
Λ_M
c_2
Λ_L
L_L
c_1

FIGURE 8-5

Origin at L, with geodetic coordinates H_L, L_L, Λ_L

ℓ_1 ℓ_2-plane normal to the plumb line direction at L'

ℓ_1 axis positive east

ℓ_2 axis positive north

ℓ_3 axis positive up, along the geodetic vertical through L, and given the coordinates of a point, M, with respect to L, defined by the vector $\vec{R}_m^{\ell}$

$$\vec{R}_m^{\ell} = x_m^{\ell} \vec{\ell}_1 + y_m^{\ell} \vec{\ell}_2 + z_m^{\ell} \vec{\ell}_3. \qquad 8\text{-}10$$

Find the geocentric and geodetic coordinates of M.

The problem of finding the geocentric coordinates of M is the problem of defining the vector $\vec{\rho}_M$ from the center of the earth of M. Figure 8-5 shows that

$$\vec{\rho}_M = \vec{R}_m^{\ell} + \vec{H}_{\ell} + \vec{\rho}_{L'} \qquad 8\text{-}11$$

$\vec{R}_m^{\ell}$ vector from L to M

$\vec{H}_L$ vector from L' to L, where L' is the foot of the perpendicular from L to the earth ellipsoid

$\rho_{L'}$ vector from the center of the earth to L'.

To use equation (8-11) the vectors must be referred to a common coordinate system. For this purpose, define the Cartesian system c_i

Origin	at the earth's center C
c_1 c_2-plane	the equatorial plane
$\vec{c}_3$	along the polar axis, positive north
$\vec{c}_2$	in the plane of the reference meridian.

Then

$$\vec{R}_m{}^c = \vec{\rho}_M{}^c = \begin{bmatrix} |\rho_M| \cos \mathscr{L}_M \sin \Lambda_M \\ |\rho_M| \cos \mathscr{L}_M \cos \Lambda_M \\ |\rho_M| \sin \mathscr{L}_M \end{bmatrix} \tag{8-12}$$

$$\vec{\rho}_{L'}{}^c = \begin{bmatrix} |\rho_{L'}| \cos \mathscr{L}_{L'} \sin \Lambda_{L'} \\ |\rho_{L'}| \cos \mathscr{L}_{L'} \cos \Lambda_{L'} \\ |\rho_{L'}| \sin \mathscr{L}_{L'} \end{bmatrix} . \tag{8-13}$$

Using the notation defined in Section 9, the vector $\vec{R}_m^{\ell}$ in the c_i system is

$$(\vec{R}_m^{\ell})^c = \left[R_3 (\pi + \Lambda_L)\right] \left[R_1 (L_L - \pi/2)\right] \vec{R}_m^{\ell}. \tag{8-14}$$

In verbal terms, the transformation from the ℓ_i system to the c_i system is obtained by

(1) rotating about the ℓ_1 axis through an angle $-(\pi/2 - L_L)$, so that the rotated "3" axis is parallel to the c_3 axis.

(2) rotating about the rotated "3" axis through an angle $(\pi + \Lambda_L)$.

Similarly, since $\vec{H}_L$ is in the direction of the ℓ_3 axis:

$$\vec{H}_L{}^c = \left[R_3 (\pi + \Lambda_L)\right] \left[R_1 (L_L - \pi/2)\right] \vec{H}_L^{\ell} \tag{8-15}$$

where

$$\vec{H}_L^{\ell} = \begin{bmatrix} 0 \\ 0 \\ H_L \end{bmatrix} .$$

Combining equations (8-10) through (8-16) yields the scalar equations

$$x_m^c = \left|\rho_M\right| \cos \mathcal{L}_M \sin \Lambda_M = \left|\rho_{L'}\right| \cos \mathcal{L}_{L'} \sin \Lambda_{L'} - x_m^{\ell} \cos \Lambda_{L'}$$

$$- y_m^{\ell} \sin L_{L'} \sin \Lambda_{L'} + (z_m^{\ell} + H_L) \cos L_{L'} \sin \Lambda_{L'} \qquad 8\text{-}17$$

$$y_m^c = \left|\rho_M\right| \cos \mathcal{L}_M \cos \Lambda_M = \left|\rho_{L'}\right| \cos \mathcal{L}_{L'} \cos \Lambda_{L'} + x_m^{\ell} \sin \Lambda_{L'}$$

$$-y_m^{\ell} \sin L_{L'} \cos \Lambda_{L'} + (z_m^{\ell} + H_L) \cos L_{L'} \cos \Lambda_{L'} \qquad 8\text{-}18$$

$$z_m^c = \left|\rho_M\right| \sin \mathcal{L}_M = \left|\rho_{L'}\right| \sin \mathcal{L}_{L'} + y_m^{\ell} \cos L_{L'} + (z_m^{\ell} + H_L) \sin L_{L'} \,. \qquad 8\text{-}19$$

Then by taking the sum of the squares of equations (8-17), (8-18) and (8-19)

$$\rho_M = \Big[(\rho_{L'}^2) + (x_m^{\ell})^2 + (y_m^{\ell})^2 + (z_m^{\ell} + H_L)^2$$

$$-2\rho_{L'} \left\{ y_m^{\ell} \sin (L_{L'} - \mathcal{L}_{L'}) - (z_m^{\ell} + H_L) \cos (L_{L'} - \mathcal{L}_{L'}) \right\} \Big]^{1/2} \qquad 8\text{-}20$$

solving equation (8-19) for $\sin \mathcal{L}_M$:

$$\sin \mathcal{L}_M = \frac{z_m^c}{\rho_M}$$

$$\sin \mathcal{L}_M = \frac{\rho_{L'}}{\rho_M} \sin \mathcal{L}_{L'} + \frac{y_m^{\ell}}{\rho_M} \cos L_{L'} + \frac{(z_m^{\ell} + H_L)}{\rho_M} \sin L_{L'} \qquad 8\text{-}21$$

and since $(-90° \leq \mathcal{L}_M \leq 90°)$

$$\cos \mathcal{L}_M = \sqrt{1 - \sin^2 \mathcal{L}_M} \,. \qquad 8\text{-}22$$

Multiplying (8-17) by $\cos \Lambda_{L'}$ and (8-18) by $\sin \Lambda_{L'}$ and subtracting yields:

$$\sin (\Lambda_M - \Lambda_L) = \frac{x_m^c \cos \Lambda_L - y_m^c \sin \Lambda_L}{\rho_M \cos \mathcal{L}_M}$$

$$\sin(\Lambda_M - \Lambda_L) = \frac{x_m^{\ell}}{\rho_M} \frac{1}{\cos \mathcal{L}_M} . \qquad 8\text{-}23$$

Multiplying (8-17) by sin $\Lambda_{L'}$ and (8-18) by cos $\Lambda_{L'}$ and adding yields:

$$\cos(\Lambda_M - \Lambda_L) = \frac{x_m^c \sin\Lambda_L + y_m^c \cos\Lambda_L}{\rho_M \cos \mathcal{L}_M}$$

$$\cos(\Lambda_M - \Lambda_L) = \frac{1}{\cos \mathcal{L}_M} \frac{\rho_M}{R_M} \cos \mathcal{L}_{L'} - \frac{y_m^{\ell}}{R_M} \sin L_{L'}$$

$$+ \frac{(z_m^{\ell} + H_L)}{R_M} \cos L_{L'} . \qquad 8\text{-}24$$

Equations (8-20) through (8-24) give the geocentric coordinates of the point, M ρ_M, $\mathcal{L}_M$, Λ_M. To find the geodetic coordinates H_M, L_M, Λ_M the best approach is an approximation, since exact solution requires solving a cubic equation. Compute a first approximation to $\rho_{M'}$, by using $\mathcal{L}_M$ in equation (8-1). This yields the radius to the surface at $\mathcal{L}_M$

$$(\rho_{M'})_1 = \frac{a}{\sqrt{1+k \sin^2 \mathcal{L}_M}} . \qquad 8\text{-}25$$

Then since $\delta \mathcal{L}_M$ is very small

$$(H_M)_1 = \rho_M - (\rho_{M'})_1 . \qquad 8\text{-}26$$

This value of (H_M) is accurate to $1:10^5$.

$$(\delta L_{M'})_1 = \tan^{-1} \left(\frac{k \sin \mathcal{L}_M \cos \mathcal{L}_M}{1+k \sin^2 \mathcal{L}_M} \right) . \qquad 8\text{-}27$$

Using the law of sines in the triangle MCM' in Figure 8-3 yields

$$(\delta \mathcal{L}_M)_1 = \sin^{-1} \left[\frac{(H_M)_1 \sin (\delta L_{M'})_1}{R_M} \right] \qquad 8\text{-}28$$

$$(\alpha_{M'})_1 = \alpha_M - (\delta\alpha_M)_1 \tag{8-29}$$

$$(\delta L_{M'})_2 = \frac{k \sin(\alpha_{M'})_1 \cos(\alpha_{M'})_1}{1+k \sin^2(\alpha_{M'})_1} \tag{8-30}$$

$$(\delta\alpha_M)_2 = \sin^{-1}\left(\frac{(H_M)_1 \sin(\delta L_{M'})_2}{R_M}\right) \tag{8-31}$$

$$L_M = \alpha_M + (\delta L_{M'})_2 + (\delta\alpha_M)_2 . \tag{8-32}$$

8.7 The problem of transformation stated in 8.1.2 above is restated here: Given a Cartesian coordinate system ℓ_i defined as 8.6 and a system m_i with

Origin	at the point, M, as defined in 8.6
$\vec{m}_3$	in the direction of the vector, $\vec{\rho}_M$ from the center of the earth to M
$\vec{m}_2$	normal to $\vec{m}_3$, in the plane of the meridian of M
$\vec{m}_1$	normal to $\vec{m}_2$ and $\vec{m}_3$, positive east.

Find the transformation between ℓ_i and m_i.

Figure 8-6 illustrates the relationships. The equation of transformation of any vector $\vec{V}$ from m_i to ℓ_i system is

$$\vec{V}^{\ell} = \left[(R_1(\pi/2 - L_L)\right]\left[R_3(\Lambda_M - \Lambda_L)\right]\left[R_1(\alpha_M - \pi/2)\right]\vec{V}^m. \tag{8-33}$$

In verbal terms, the transformation is obtained by

(1) rotating about the m_1 axis through an angle $-(\pi/2 - \alpha_M)$

(2) rotating about the rotated "3" axis through an angle $(\Lambda_M - \Lambda_L)$, so that the rotated "2" axis and "3" axis are in the meridian plane of L

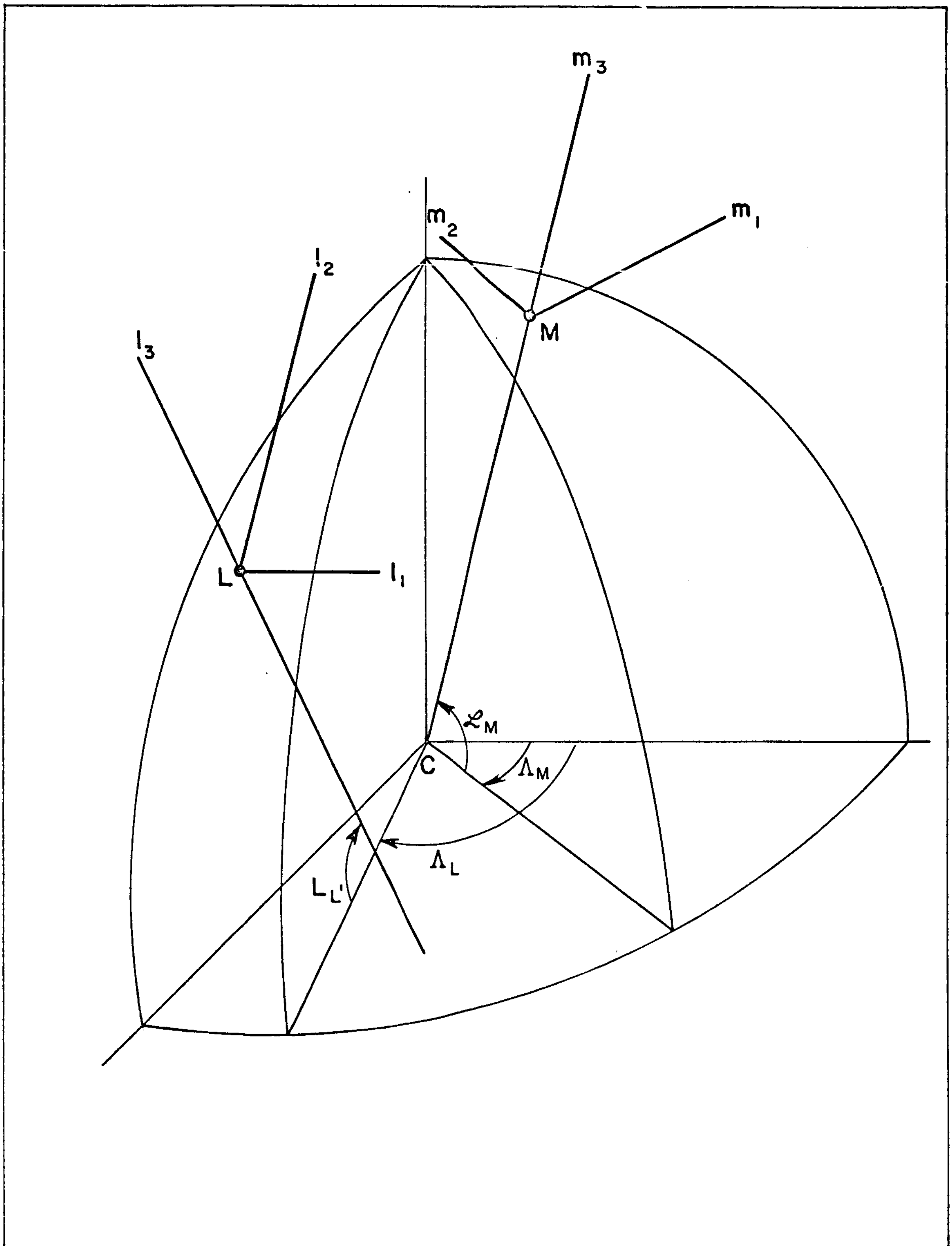


FIGURE 8-6

(3) rotating about the "1" axis through an angle $(\pi/2 - L_L)$.
Equation 8-33 is written

$$\vec{V}^{\ell} = \left[a_{ij}\right] \vec{V}^{m} \qquad 8\text{-}34$$

and the terms of $\left[a_{ij}\right]$ can be obtained by substituting into (8-33) the forms of the elementary rotations defined in (9-8), (9-9) and (9-10), with $\Lambda_M - \Lambda_L = \Delta\Lambda$.

$$a_{11} = \cos \Delta\Lambda$$

$$a_{12} = \sin \Delta\Lambda \ \sin \ell_M$$

$$a_{13} = -\sin \Delta\Lambda \ \cos \ell_M$$

$$a_{21} = -\sin L_L \ \sin \Delta\Lambda$$

$$a_{22} = \cos L_L \ \cos \ell_M + \sin L_L \ \cos \Delta\Lambda \ \sin \ell_M$$

$$a_{23} = \cos L_L \ \sin \ell_M - \sin L_L \ \cos \Delta\Lambda \ \cos \ell_M$$

$$a_{31} = \cos L_L \ \sin \Delta\Lambda$$

$$a_{32} = \sin L_L \ \cos \ell_M - \cos L_L \ \cos \Delta\Lambda \ \sin \ell_M$$

$$a_{33} = \sin L_L \ \sin \ell_M + \cos L_L \ \cos \Delta\Lambda \ \cos \ell_M . \qquad 8\text{-}35$$

The matrix $\left[a_{ij}\right]$ defined by the nine equations (8-35) is the matrix such that

$$\vec{V}^{\ell} = \left[a_{ij}\right] \vec{V}^{m} .$$

8.8 The problem of transformation stated in 8.1.3 is restated here:

Given a Cartesian coordinate system ℓ_i defined as in 8.6 and a Cartesian system t_i defined as follows:

Origin	at T with geodetic coordinates (H_T, L_T, Λ_T)
t_1 t_2-plane	normal to the plumb line direction at T
t_1	axis positive east
t_2	axis positive north
t_3	axis positive up, along the geodetic vertical through T.

<u>Given</u> $x_m{}^\ell$, $y_m{}^\ell$, $z_m{}^\ell$, the coordinates of a point, M, with respect to L. <u>Find</u> $x_m{}^t$, $y_m{}^t$, $z_m{}^t$ the coordinates of M with respect to T.

Figure 8-7 shows the relationships.

$$\vec{R}_m{}^t = -\vec{H}_T - \vec{\rho}_{T'} + \vec{\rho}_{L'} + \vec{H}_L + \vec{R}_m{}^\ell \qquad \text{8-35}$$

$$\vec{H}_T{}^t = \begin{bmatrix} 0 \\ 0 \\ H_T \end{bmatrix} \qquad \text{8-36}$$

$$\vec{\rho}_{T'}{}^t = \left[R_1\ (-\delta L_T)\right] \begin{bmatrix} 0 \\ 0 \\ \rho_{T'} \end{bmatrix} \qquad \text{8-37}$$

$$\vec{\rho}_{L'}{}^t = \left[R_1\ (\pi/2 - L_T)\right]\left[R_3\ (-\Lambda_T + \Lambda_L)\right]\left[R_1\ \alpha_L - \pi/2)\right] \begin{bmatrix} 0 \\ 0 \\ \rho_{L'} \end{bmatrix} \qquad \text{8-38}$$

$$\vec{H}_L{}^t = \left[R_1\ (\pi/2 - L_T)\right]\left[R_3\ (-\Lambda_T + \Lambda_L)\right]\left[R_1\ (L_L - \pi/2)\right] \begin{bmatrix} 0 \\ 0 \\ H_L \end{bmatrix} \qquad \text{8-39}$$

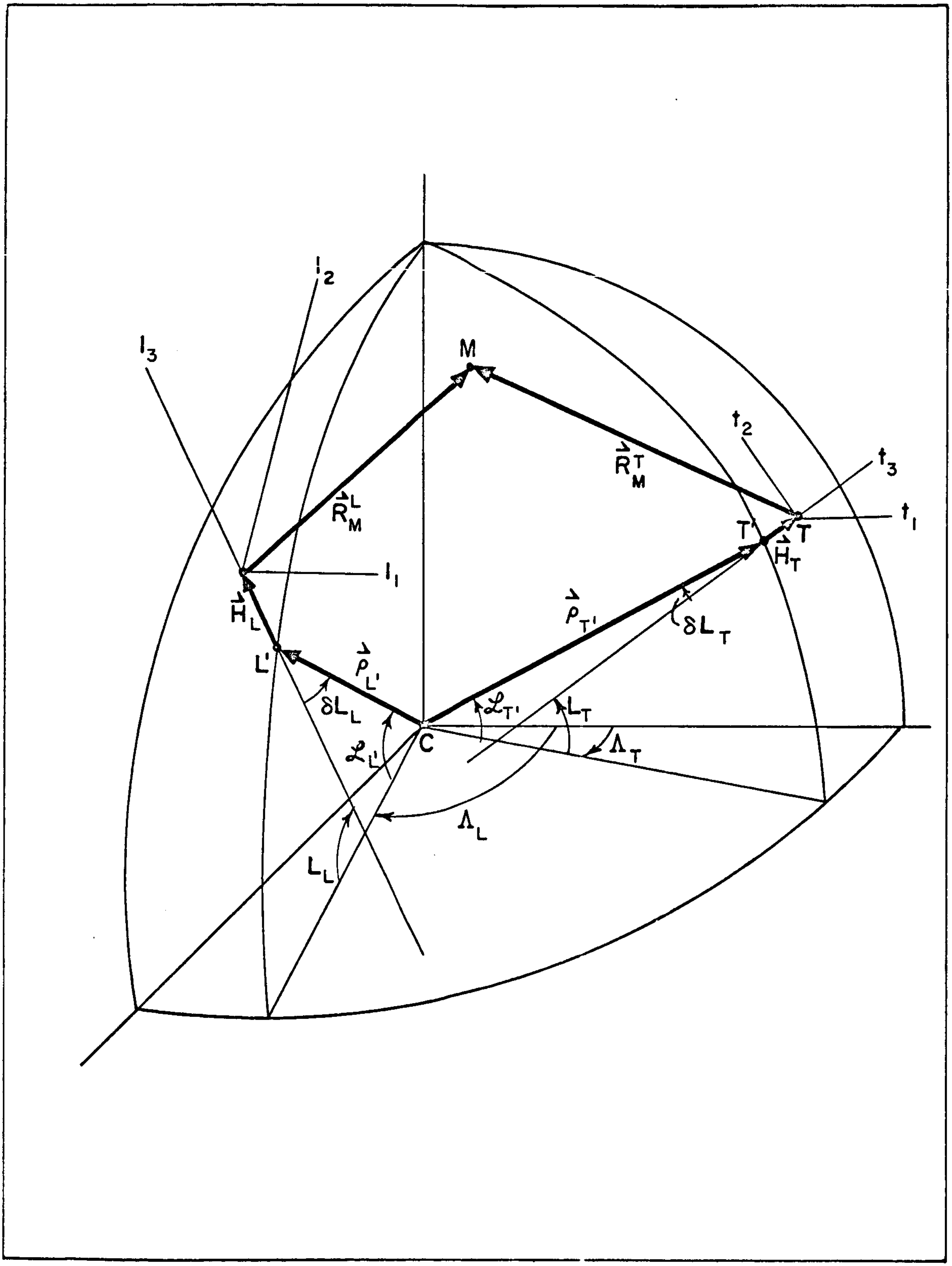


FIGURE 8-7

$$(\vec{R}_m{}^{\ell})^t = \left[R_1\left(\pi/2 - L_T\right)\right]\left[R_3\left(-\Lambda_T + \Lambda_L\right)\right]\left[R_1\left(L_L - \pi/2\right)\right]\begin{bmatrix} x_m{}^{\ell} \\ y_m{}^{\ell} \\ z_m{}^{\ell} \end{bmatrix} . \tag{8-40}$$

Using equations (8-36) through (8-40) in (8-35), and using the definitions of Section 9 results in:

$$\begin{pmatrix} x_m{}^t \\ y_m{}^t \\ z_m{}^t \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{pmatrix} x_m{}^{\ell} \\ y_m{}^{\ell} \\ z_m{}^{\ell} \end{pmatrix} . \tag{8-41}$$

In (8-41), the elements are:

$$b_{11} = \cos\left(\Lambda_T - \Lambda_L\right)$$

$$b_{12} = -\sin\left(\Lambda_T - \Lambda_L\right) \sin L_L$$

$$b_{13} = \sin\left(\Lambda_T - \Lambda_L\right) \cos L_L$$

$$b_{21} = \sin L_T \sin\left(\Lambda_T - \Lambda_L\right)$$

$$b_{22} = \cos L_T \cos L_L + \sin L_T \cos\left(\Lambda_T - \Lambda_L\right) \sin L_L$$

$$b_{23} = \cos L_T \sin L_L - \sin L_T \cos\left(\Lambda_T - \Lambda_L\right) \cos L_L$$

$$b_{31} = -\cos L_T \sin\left(\Lambda_T - \Lambda_L\right)$$

$$b_{32} = \sin L_T \cos L_L - \cos L_T \cos\left(\Lambda_T - \Lambda_L\right) \cos L_L$$

$$b_{33} = \sin L_T \sin L_L + \cos L_T \cos\left(\Lambda_T - \Lambda_L\right) \cos L_L . \tag{8-42}$$

$$k_1 = -b_{12}\ \rho_L \sin\ \delta L_L + b_{13}\ (H_L + \rho_L \cos\ \delta L_L)$$

$$k_2 = -\ \rho_T \sin\ \delta L_T - b_{22}\ \rho_L \sin\ \delta L_L + b_{23}\ (H_L + \rho_L \cos\ \delta L_L)$$

$$k_3 = -H_T - \rho_T \cos\ \delta L_T - b_{32}\ \rho_L \sin\ \delta L_L + b_{33}\ (H_L + \rho_L \cos\ \delta L_L). \qquad 8\text{-}43$$

The nine elements of (8-42) give the rotation, and the three elements of (8-43) give the translation for transforming from the ℓ_1 system to the t_i system.

8.9 The differential equations of Program 5 are computed in a system f_i, with

Origin	at launcher
$f_1\ f_2$-plane	coincident with $\ell_1\ \ell_2$-plane
f_1	along firing azimuth
f_3	along ℓ_3 axis.

The transformation from ℓ_i to f_i is

$$\vec{V}^f = \left[R_3\ (\alpha_f - 90°)\right]\ \vec{V}^\ell$$

αf azimuth of firing, clockwise from north.

9. Vector and Matrix Notation and Definitions

9.1 The notation defined in this section is intended to provide a systematic and relatively compact means of writing vector equations which involve coordinate transformations. The key to the notation is the use of subscripts and superscripts to define the coordinate system used. Thus, the basic symbol assigned to a vector remains the same, even though the coordinates are transformed to a different system.*

With one exception, the definitions given below are repetitions of standard definitions in elementary vector and matrix theory. They are repeated here to illustrate the notation. The exception is the definition of the derivative of a vector, and the resulting derivation of the equation for the derivative of a vector in a rotating coordinate system.

9.1.1 In discussion of Cartesian coordinate systems each of the three axes will be referred to by an identifying letter, characterizing the coordinate system, with a subscript number, 1, 2 or 3, identifying the axis. For example, the axes of a body based system will be called b_1, b_2, b_3; the axes of a ground based system, g_1, g_2, g_3. The systems will be referred to as the b_i or the g_i system. The unit vectors directed along the three axes of a system will be designated by the axis identifiers with a vector symbol. For example, $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$; $\vec{g}_1$, $\vec{g}_2$, $\vec{g}_3$. The three axes of a system are so defined

*The notation is adopted from that used in Reference 5.

that they constitute a right handed system. The conditions for a right handed system are:

$$\vec{g}_1 \times \vec{g}_2 = \vec{g}_3; \ \vec{g}_2 \times \vec{g}_1 = -\vec{g}_3$$

$$\vec{g}_2 \times \vec{g}_3 = \vec{g}_1; \ \vec{g}_3 \times \vec{g}_2 = -\vec{g}_1$$

$$\vec{g}_3 \times \vec{g}_1 = \vec{g}_2; \ \vec{g}_1 \times \vec{g}_3 = -\vec{g}_2.$$

<u>9.1.2</u> A vector which is expressed in terms of components in a given system is identified by using the identifying letter of that system as a superscript. For example, the vector $\vec{V}$ expressed in the body coordinate system is $\vec{V}^b$. The scalar components of the vector in the system referred to are identified by the vector identifier, with the coordinate system identifier as a superscript and the axis identifier as a subscript. For example:

$$\vec{V}^b = V_1{}^b \, \vec{b}_1 + V_2{}^b \, \vec{b}_2 + V_3{}^b \, \vec{b}_3. \qquad 9\text{-}2$$

In the special case of a radius vector from the origin of a system to a point the symbol used will be $\vec{R}$ with a superscript identifying the coordinate system, and a subscript identifying the point. The components of such a radius vector will be written x, y, z, with the same subscript and superscript.

For example $\vec{R}_m{}^g$ will designate the radius vector from the origin of the g_i system to the point M, and is written in component form:

$$\vec{R}_m{}^g = x_m{}^g \, \vec{g}_1 + y_m{}^g \, \vec{g}_2 + z_m{}^g \, \vec{g}_3.$$

Vectors will also be written as column matrices, i.e.

$$\vec{V}^s = \begin{bmatrix} V_1{}^s \\ V_2{}^s \\ V_3{}^s \end{bmatrix}$$

$$\vec{R}_m{}^g = \begin{bmatrix} x_m{}^g \\ y_m{}^g \\ z_m{}^g \end{bmatrix} .$$

9.1.3 The vector identifiers are upper case, and the corresponding unit vectors lower case. The unit vector in the direction of a vector $\vec{V}$ is $\vec{v}$, and the magnitude is $|V|$, so that:

$$\vec{V} = |V| \, \vec{v} \qquad 9\text{-}3$$

and

$$\vec{V}^b = |V| \, \vec{v}^b$$

$$\vec{V}^b = |V| \, (v_1{}^b \, \vec{b}_1 + v_2{}^b \, \vec{b}_2 + v_3{}^b \, \vec{b}_3). \qquad 9\text{-}4$$

The components v_1, v_2, v_3, of the unit vector $\vec{v}$ in a given system are the direction cosines of $\vec{V}$ in that system.

9.2 Coordinate Transformation Notation and Conventions

9.2.1 If r_i and s_i are any two Cartesian coordinate systems, and if $\vec{V}$ is any vector, the relation between the components of $\vec{V}$ in the r_i system and the components of $\vec{V}$ in the s_i system is written:*

*Reference 6, Chapter 4.

$$\begin{bmatrix} V_1{}^s \\ V_2{}^s \\ V_3{}^s \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} V_1{}^r \\ V_2{}^r \\ V_3{}^r \end{bmatrix}. \qquad 9\text{-}5$$

In this equation, a_{ij} is the cosine of the angle between $\vec{s}_i$ and $\vec{r}_j$. The elements, a_{i1}, a_{i2}, a_{i3}, of the i^{th} row of the matrix are the direction cosines of the s_i axis in the r_1, r_2, r_3 system. The elements, a_{1j}, a_{2j}, a_{3j} of the j^{th} column of the matrix are the direction cosines of the r_j axis in the s_1, s_2, s_3 system.

The inverse relationship to (9-5) is:

$$\begin{bmatrix} V_1{}^r \\ V_2{}^r \\ V_3{}^r \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} V_1{}^s \\ V_2{}^s \\ V_3{}^s \end{bmatrix}. \qquad 9\text{-}6$$

The matrix of the inverse transformation is the transpose of the original matrix.

The relationship (9-5) also holds for the unit vectors of the two systems:

$$\begin{bmatrix} \vec{s}_1 \\ \vec{s}_2 \\ \vec{s}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix}. \qquad 9\text{-}7$$

The matrix of equations (9-5) and (9-7) is called a "direction cosine matrix", an "orthogonal transformation matrix" or simply a "rotation matrix". For ease of writing, equation (9-5) is written

$$\vec{V}^s = \left[a_{ij}\right] \vec{V}^r$$

and in general, a matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is written $\left[a_{ij}\right]$.

The element in the i^{th} row and j^{th} column is written a_{ij}.

The inverse matrix

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

is written $\left[a_{ji}\right]$.

9.2.2 Several facts about matrices such as $\left[a_{ij}\right]$ are important:

(a) The "orthogonality conditions" are

$$\sum_{i=1}^{3} a_{ij}^2 = \sum_{j=1}^{3} a_{ij}^2 = 1$$

$$\sum_{\substack{i=1 \\ j \neq k}}^{3} a_{ij}\, a_{ik} = \sum_{\substack{j=1 \\ i \neq k}}^{3} a_{ij}\, a_{kj} = 0 .$$

(b) The determinant of the matrix $\left[a_{ij}\right]$ equals 1, if $\left[a_{ij}\right]$ is a transformation between two right handed systems.

(c) The co-factor of any element a_{ij}, in the determinant of the matrix $\left[a_{ij}\right]$, is equal to a_{ij}.

(d) The identity transformation matrix is designated by $\begin{bmatrix} 1 \end{bmatrix}$;

$$\begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} a_{ij} \end{bmatrix} \begin{bmatrix} a_{ji} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}.$$

(e) If $\begin{bmatrix} a_{ij} \end{bmatrix}$ and $\begin{bmatrix} c_{ij} \end{bmatrix}$ are orthogonal transformation matrices, there exists an orthogonal transformation matrix $\begin{bmatrix} b_{ij} \end{bmatrix}$ such that

$$\begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} b_{ij} \end{bmatrix} \begin{bmatrix} c_{ij} \end{bmatrix} .$$

<u>9.2.3</u> If r_i, s_i and t_i are any three distinct Cartesian systems and $\vec{V}$ is any vector, the relationships can be written:

$$\vec{V}^s = \begin{bmatrix} d_{ij} \end{bmatrix} \vec{V}^r$$
$$\vec{V}^t = \begin{bmatrix} e_{ij} \end{bmatrix} \vec{V}^s$$
$$\vec{V}^t = \begin{bmatrix} f_{ij} \end{bmatrix} \vec{V}^r$$

then

$$\begin{bmatrix} f_{ij} \end{bmatrix} = \begin{bmatrix} e_{ij} \end{bmatrix} \begin{bmatrix} d_{ij} \end{bmatrix} .$$

That is: the matrix defining the result of two successive rotations is obtained by taking the product of the matrices defining the individual rotations, with the matrix of the last rotation on the left. Since matrix multiplication is not commutative, the matrices must be multiplied in the same order as the rotations are considered.

9.2.4 The transformation of the coordinates of a point M from one system to another involves a translation and a rotation

$$\vec{R}_m{}^s = \left[a_{ij}\right] \vec{R}_m{}^r + \vec{R}_r{}^s$$

$$= \left[a_{ij}\right] (\vec{R}_m{}^r - \vec{R}_s{}^r)$$

$\vec{R}_m{}^s$ the vector from S to M expressed in the s_i system

$\vec{R}_m{}^r$ the vector from R to M expressed in the r_i system

$\vec{R}_r{}^s$ the vector from S to R in the s_i system

$\vec{R}_s{}^r$ the vector from R to S in the r_i system.

9.3 Elementary Rotations

In many applications, the 3 x 3 matrix $\left[a_{ij}\right]$ is difficult to obtain directly. For this reason, we consider elementary rotation matrices. An elementary rotation is here defined as one in which one of the axes remains fixed, and the other two axes rotate about the fixed axis. Any possible elementary rotation can be expressed in terms of one of the three matrix operators defined below.

9.3.1 A positive rotative rotation about the "1" axis is defined as one by which the "2" axis rotates toward the "3" axis. If the axes s_i are defined as the axes obtained by rotating the axes r_2, r_3 about r_1, through an angle Θ as in Figure (9-1a) then:

$$
\begin{bmatrix} V_1{}^s \\ V_2{}^s \\ V_3{}^s \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V_1{}^r \\ V_2{}^r \\ V_3{}^r \end{bmatrix} . \qquad 9\text{-}8
$$

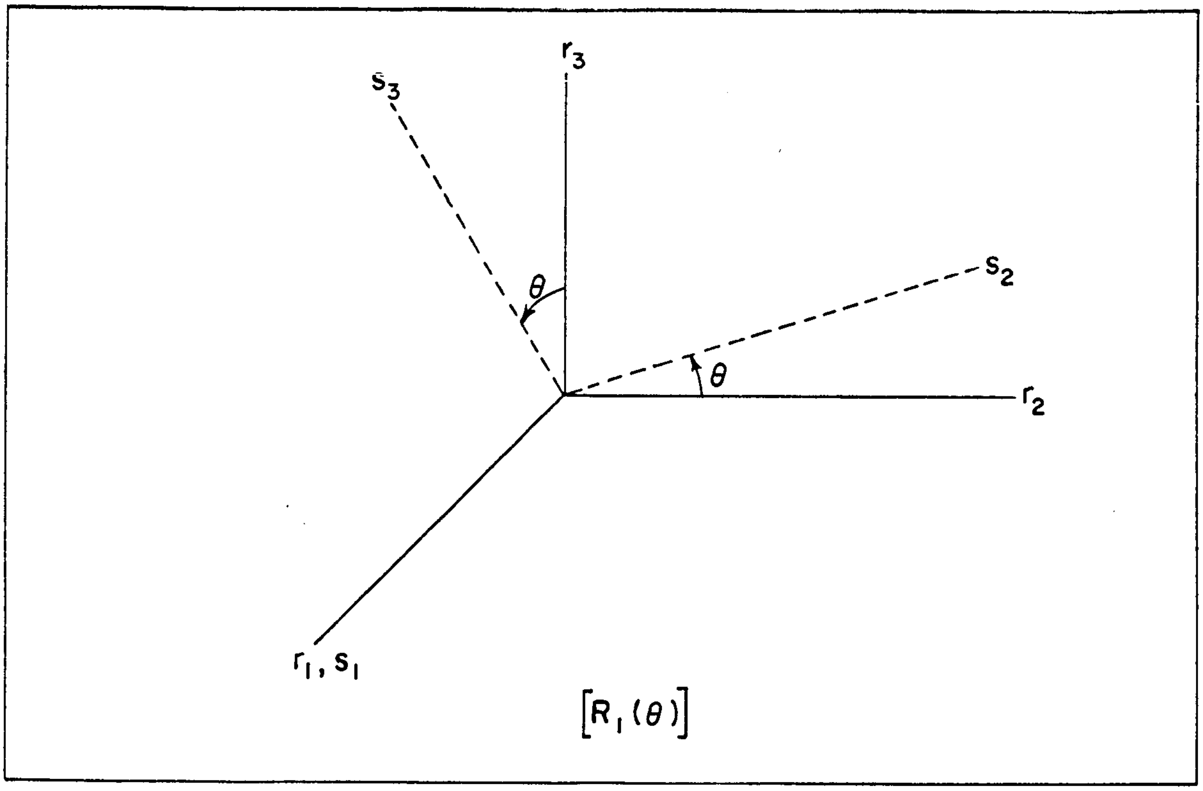


FIGURE 9-1a. Elementary Rotation about $\dot{r}_1$ Axis

All positive rotations about the "1" axis yield matrices of the form

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos & \sin \\ 0 & -\sin & \cos \end{bmatrix} .
$$

This form will be abbreviated to $[R_1]$ and equation (9-8) written

$$
\vec{V}^s = [R_1(\theta)]\ \vec{V}^r.
$$

9.3.2 A positive rotation about the "2" axis is defined as one by which the "3" axis rotates toward the "1" axis. If axes t_i are defined as those obtained by rotating s_3, s_1 about s_2, through an angle ϕ as in Figure 9-1b then

$$\begin{bmatrix} V_1^t \\ V_2^t \\ V_3^t \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} V_1^s \\ V_2^s \\ V_3^s \end{bmatrix} . \qquad 9\text{-}9$$

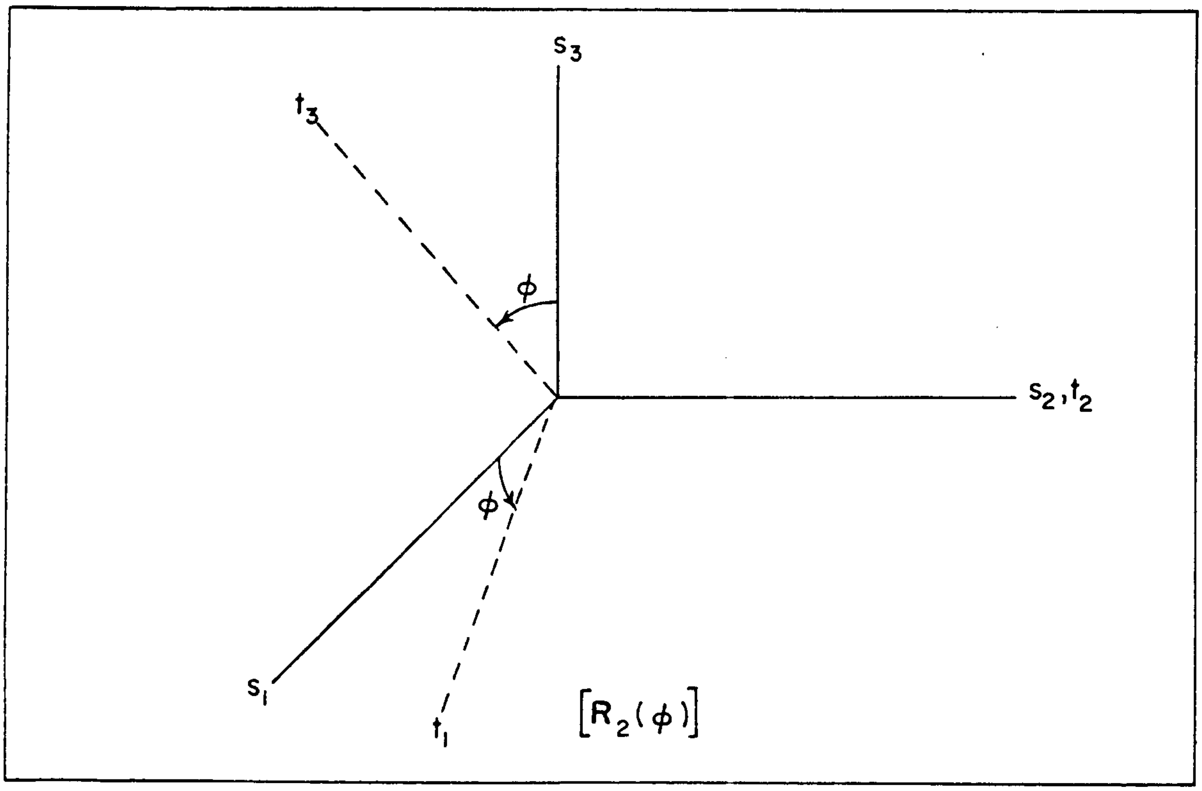


FIGURE 9-1b. Elementary Rotation about s_2 Axis

All positive rotations about the "2" axis yield matrices of the form

$$\begin{bmatrix} R_2 \end{bmatrix} = \begin{bmatrix} \cos & 0 & -\sin \\ 0 & 1 & 0 \\ \sin & 0 & \cos \end{bmatrix}$$

and equation (9-9) is written

$$\vec{V}^t = \left[R_2\ (\phi)\right]\ \vec{V}^s.$$

9.3.3 A positive rotation about the "3" axis is defined as one by which the "1" axis rotates toward the "2" axis. If axes u_i are defined as those obtained by rotating t_1, t_2 about t_3, through an angle ψ as in Figure 9-1c then

$$\begin{bmatrix} V_1{}^u \\ V_2{}^u \\ V_3{}^u \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1{}^t \\ V_2{}^t \\ V_3{}^t \end{bmatrix}. \qquad 9\text{-}10$$

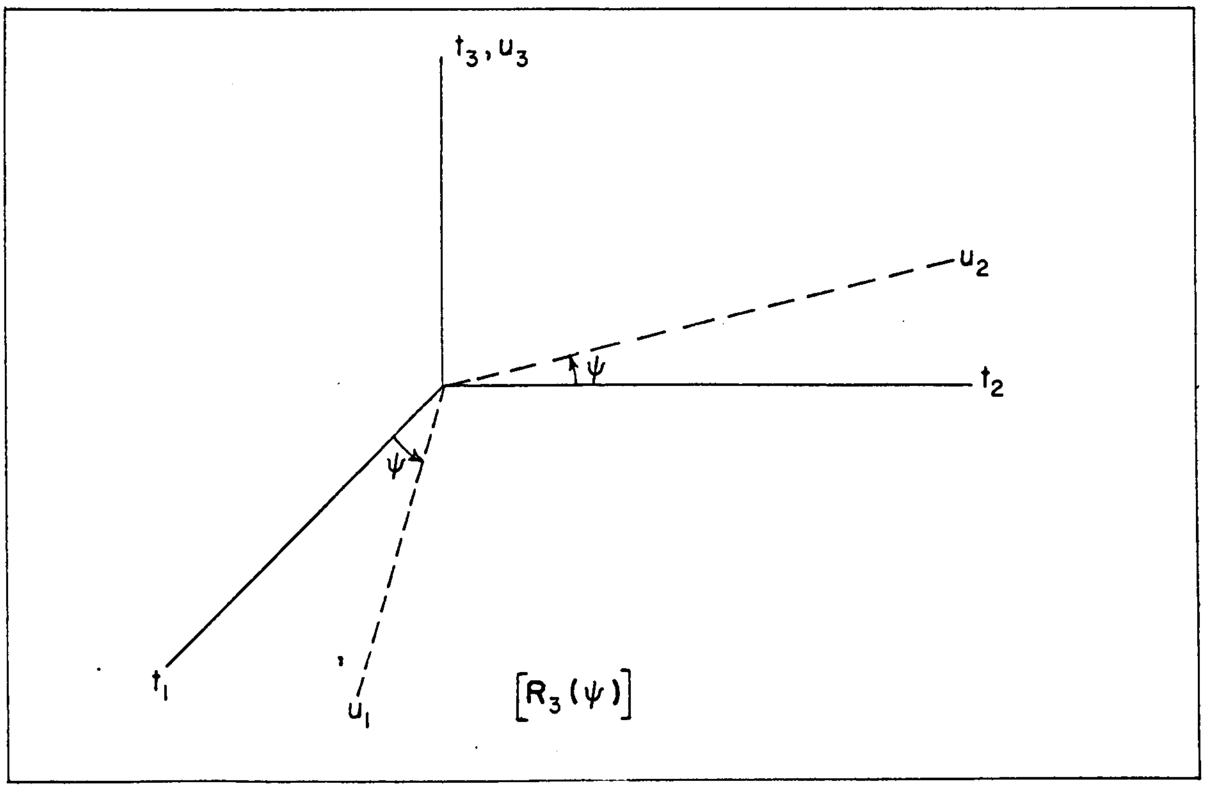


FIGURE 9-1c. Elementary Rotation about t_3 about Axis

All positive rotations about the "3" axis yield matrices of the form

$$\left[R_3\right] = \begin{bmatrix} \cos & \sin & 0 \\ -\sin & \cos & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and equation (4-8) is written

$$\vec{V}^u = \left[R_3\ (\psi)\right] \vec{V}^t.$$

9.3.4 The form of the elementary rotation matrix is, of course, independent of the symbol used to designate the angle. For example, a rotation about the "1" axis through an angle α, as shown in Figure 9-2a would be represented by the matrix:

$$\left[R_1\ (\alpha)\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}.$$

If the angle used in the rotation is not positive in the sense defined in 9.3.1; for example if the rotation is through an angle β about r_1 from r_3 toward r_2, as shown in Figure 9-2b, then the rotation matrix is:

$$R_1\ (-\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\beta) & \sin(-\beta) \\ 0 & -\sin(-\beta) & \cos(-\beta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}.$$

If the angle used is not defined as the angle between the r_2 axis and the s_2 axis, the rotation can still be expressed as such an angle. For example, in Figure 9-2c, the angle λ is the angle between the s_2 axis and the r_3 axis, positive from r_3 toward s_2. This rotation is written:

$$\left[R_1(\lambda - \Pi/2)\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\lambda - \Pi/2) & \sin(\lambda - \Pi/2) \\ 0 & -\sin(\lambda - \Pi/2) & \cos(\lambda - \Pi/2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\lambda & -\cos\lambda \\ 0 & \cos\lambda & \sin\lambda \end{bmatrix}.$$

9.3.5 By using the convention that all elementary rotations will be written using the operators $\left[R_1(\ )\right]$, $\left[R_2(\ )\right]$, $\left[R_3(\ )\right]$ much confusion can be avoided. All rotation angles can be expressed in terms of positive rotations about one of the three axes, as defined above.

9.4 The result of the three rotations described by equations (9-8), (9-9) and (9-10) is written

$$\vec{V}^u = \left[R_3(\psi)\right]\left[R_2(\phi)\right]\left[R_1(\theta)\right]\vec{V}^r \qquad 9\text{-}11$$

and by multiplying the three matrices

$$\vec{V}^u \doteq \left[a_{ij}\right]\vec{V}^r$$

where

$$\left[a_{ij}\right] = \left[R_3(\psi)\right]\left[R_2(\phi)\right]\left[R_1(\theta)\right]. \qquad 9\text{-}12$$

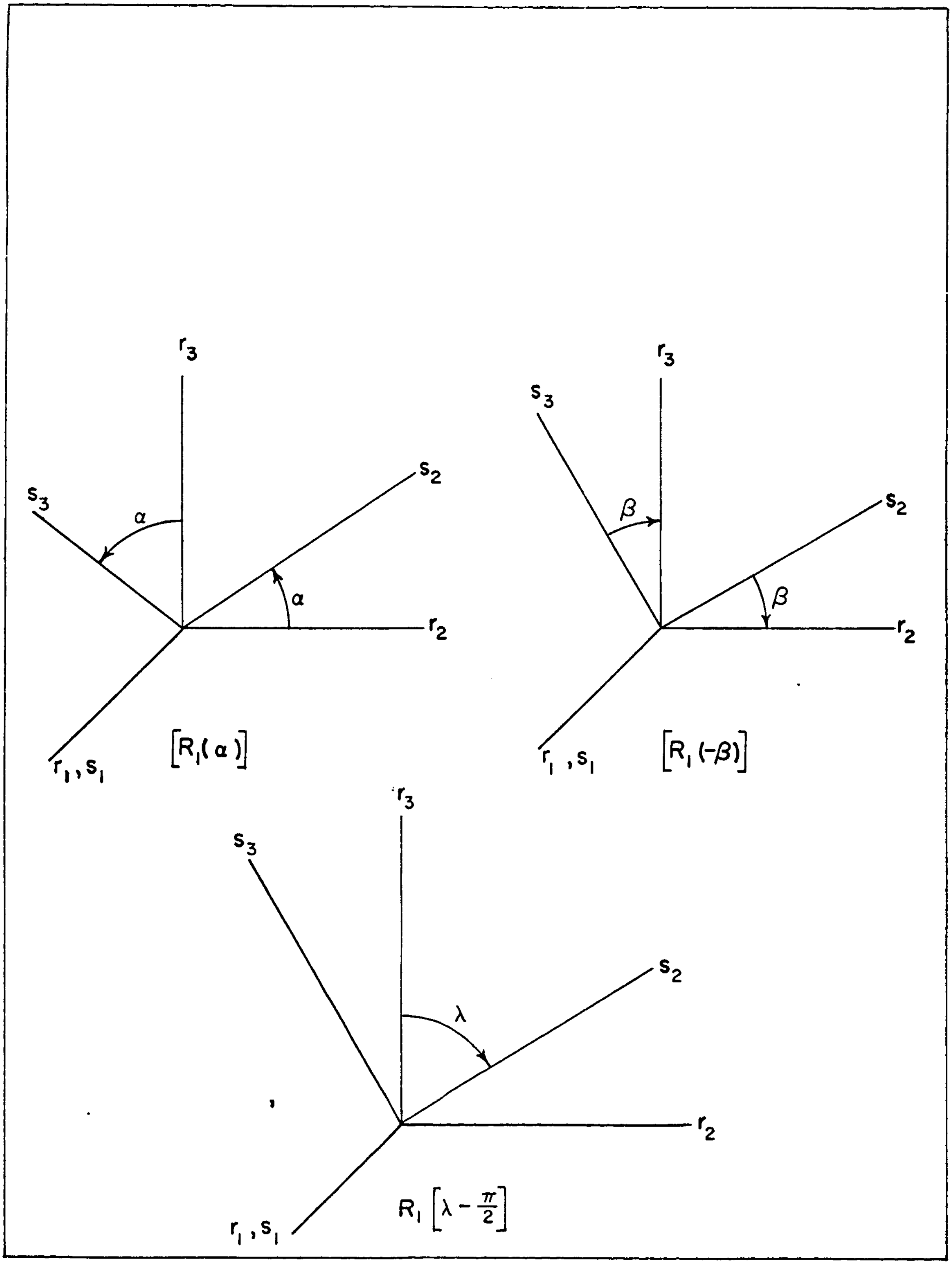

r_3
s_3
α
s_2
α
r_2
r_1, s_1
$[R_1(\alpha)]$
r_3
s_3
β
s_2
β
r_2
r_1, s_1
$[R_1(-\beta)]$
r_3
s_3
s_2
λ
r_2
r_1, s_1
$R_1\left[\lambda - \frac{\pi}{2}\right]$

FIGURE 9-2

Any matrix $\left[a_{ij}\right]$ defines a unique relationship between two coordinate systems, and depends only on the relative orientation of the systems. The series of rotations as defined in (9-8), (9-9), (9-10) may be defined in the conditions of the problem, or may be a mathematical artifice to enable us to see the transformation one step at a time, in two dimensional representation. These elementary rotations define the transformation in terms of angles, which are sometimes easier to work with than direction cosines.

The matrix $\left[a_{ij}\right]$ was defined in terms of three elementary rotations, as shown by (9-11). The rotation at the right was taken first, so that this was a sequence R_1, R_2, R_3. If the sequence had been different, for example R_1, R_3, R_2 then the resulting matrix product $\left[R_2\ (\phi)\right] \left[R_3\ (\psi)\right] \left[R_1\ (\theta)\right]$ would not equal the $\left[a_{ij}\right]$ obtained before. However, it would be possible to find three angles, which we can call ψ', ϕ', θ' such that

$$\left[a_{ij}\right] = \left[R_2\ (\phi')\right] \left[R_3\ (\psi')\right] \left[R_1\ (\theta')\right].$$

In fact for any given direction cosine matrix $\left[a_{ij}\right]$ and for any of the sequences of rotation listed below it is possible to find a set of three angles such that the resulting matrix product is equal to $\left[a_{ij}\right]$.

R_1, R_2, R_3

R_1, R_3, R_2

R_2, R_1, R_3

R_2, R_3, R_1

R_3, R_1, R_2

R_3, R_2, R_1 9-13

$$R_1, R_2, R_1$$

$$R_1, R_3, R_1$$

$$R_2, R_1, R_2$$

$$R_2, R_3, R_2$$

$$R_3, R_1, R_3$$

$$R_3, R_2, R_3 \qquad 9\text{-}14$$

will be equal to the given $\left[a_{ij}\right]$. The rotations (9-13) are called nonrepetitive sequences; the sequences (9-14) are repetitive. It is possible to express any problem involving coordinate transformations in terms of elementary rotations in any one of the twelve ways.

The choice of sequence may depend on one or several of the factors listed below:

(a) Actual knowledge of rotations involved, as implied by the geometry or by mechanical systems involved

(b) The physical laws involved, which make the computations much simpler and the output much more meaningful in one system of angles than in any other

(c) The output data from the problem which may be required to be in a given set of angles.

For example:

(a) In reduction of ballistic camera data, it is convenient to transform data from an "object space" to an "image space". If the camera is mounted on a precision three-axis mount which rotates first in azimuth, then in elevation, then in tilt, the angles used would be those measured by the scales on the mount and the angle sequence used to describe the transformation would use the same order of rotation as do the three axes of the camera.

(b) In the classical solution* of the dynamics of a force free rigid body a transformation between inertial axes i_i and body fixed axes b_i is made. The sequence chosen is R_3, R_1, R_3 resulting in a transformation matrix

$$R_3 (\psi) R_1 (\theta) R_3 (\phi).$$

The inertial coordinate system is chosen with $\vec{i}_3$ directed along the angular momentum vector of the body. With this choice of coordinates, ψ is the spin angle, θ the nutation angle and ϕ the precession angle. The physical laws have a relatively simple form in terms of these angles, and the angles themselves have definite meanings. This sequence yields the so-called "Euler Angles".

9.5 Differentiation of Vectors and Matrices

The derivative of a scalar quantity with respect to time is indicated by a dot over the quantity. The derivative of a vector quantity, as used here, is the vector made up of the derivatives of the scalar components of the original vector. In this sense, differentiation is meaningful only as defined in a specified coordinate system.

*Reference 6, Chapter 4.

In the notation defined in 9.1.2

$$\dot{\vec{V}}^r = \dot{V}_1^r \vec{r}_1 + \dot{V}_2^r \vec{r}_2 + \dot{V}_3^r \vec{r}_3. \qquad 9\text{-}15$$

The derivative of $\vec{V}^s$ is:

$$\dot{\vec{V}}^s = \dot{V}_1^s \vec{s}_1 + \dot{V}_2^s \vec{s}_2 + \dot{V}_3^s \vec{s}_3. \qquad 9\text{-}16$$

Now if

$$\vec{V}^s = \left[a_{ij}\right] \vec{V}^r \qquad 9\text{-}17$$

the derivative of (9-17) is

$$\dot{\vec{V}}^s = \left[a_{ij}\right] \dot{\vec{V}}^r + \left[\dot{a}_{ij}\right] \vec{V}^r. \qquad 9\text{-}18$$

$\left[\dot{a}_{ij}\right]$ is the matrix whose elements are the derivatives of the corresponding elements in $\left[a_{ij}\right]$. By writing (9-17) as three scalar equations and taking the derivative of both sides of each equation, then rewriting as a matrix equation, we obtain (9-18).

Equation (9-18) is equivalent to the equations

$$\dot{\vec{V}}^s = \left[a_{ij}\right] \dot{\vec{V}}^r - \vec{\omega}^s \times \vec{V}^s \qquad 9\text{-}19$$

$$\dot{\vec{V}}^s = \left[a_{ij}\right] (\dot{\vec{V}}^r - \vec{\omega}^r \times \vec{V}^r) \qquad 9\text{-}20$$

where $\vec{\omega}$ is the angular velocity vector of the s_i system with respect to the r_i system.*

Equations (9-19) and (9-20) are derived on the following page.

*Reference 6, p. 133.

9.5.1 In order to simplify the matrix $\left[\dot{a}_{ij}\right]$, in equation (9-18) let

$$\left[a_{ij}\ (t)\right] = \left[b_{ij}\ (t)\right]\ \left[a_{ij}\ (t_1)\right] \tag{9-21}$$

where $\left[a_{ij}\ (t)\right]$ is the matrix $\left[a_{ij}\right]$ as a function of time, $\left[a_{ij}\ (t_1)\right]$ is the matrix $\left[a_{ij}\right]$ at arbitrary constant time, t_1, and $\left[b_{ij}\ (t)\right]$ is the matrix which makes (9-21) true. $\left[b_{ij}\ (t)\right]$ is the matrix defining the rotation of the s_i system from the time t_1 to the time t.

Now consider the derivative of a general matrix product. If

$$\left[a_{ij}\right] = \left[b_{ij}\right]\ \left[c_{ij}\right]$$

$$\left[\dot{a}_{ij}\right] = \left[\dot{b}_{ij}\right]\ \left[c_{ij}\right] + \left[b_{ij}\right]\ \left[\dot{c}_{ij}\right].$$

Taking the derivative of both sides of (9-21) yields

$$\left[\dot{a}_{ij}\ (t)\right] = \left[\dot{b}_{ij}\ (t)\right]\ \left[a_{ij}\ (t_1)\right] + \left[b_{ij}\ (t)\right]\ \left[\dot{a}_{ij}\ (t_1)\right].$$

Since $\left[a_{ij}\ (t_1)\right]$ is the value of $\left[a_{ij}\right]$ at a constant time,

$$\left[\dot{a}_{ij}\ (t_1)\right] = 0$$

and

$$\left[\dot{a}_{ij}\ (t)\right] = \left[\dot{b}_{ij}\ (t)\right]\ \left[a_{ij}\ (t_1)\right].$$

Now $\left[b_{ij}\ (t)\right]$ can be written as the product of three elementary matrices:

$$\left[b_{ij}\ (t)\right] = \left[R_1\ (\Theta)\right]\ \left[R_2\ (\phi)\right]\ \left[R_3\ (\psi)\right]\ * \tag{9-22}$$

where $\left[R_1\ (\Theta)\right]$, $\left[R_2\ (\phi)\right]$, $\left[R_3\ (\psi)\right]$ are functions of time.

*Any non-repetitive sequence can be used in the derivation. Use of a repetitive sequence invalidates equation (9-28).

ψ, θ, ϕ are the three angles which describe the rotation of the s_i system with respect to its position at time t_1. The angles are zero at t_1. Then

$$\left[\dot{b}_{ij}\ (t)\right] = \left[\dot{R}_1\right] \left[R_2\right] \left[R_3\right] + \left[R_1\right] \left[\dot{R}_2\right] \left[R_3\right] + \left[R_1\right] \left[R_2\right] \left[\dot{R}_3\right] \qquad 9\text{-}23$$

in which the arguments of the rotations are omitted for convenience. The derivatives $\left[\dot{R}_1\right]$, $\left[\dot{R}_2\right]$, $\left[\dot{R}_3\right]$ are obtained from (9-8), (9-9), (9-10)

$$\left[\dot{R}_1\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\dot{\theta}\sin\theta & \dot{\theta}\cos\theta \\ 0 & -\dot{\theta}\cos\theta & -\dot{\theta}\sin\theta \end{bmatrix} \qquad 9\text{-}24$$

$$\left[\dot{R}_2\right] = \begin{bmatrix} -\dot{\phi}\sin\phi & 0 & -\dot{\phi}\cos\phi \\ 0 & 0 & 0 \\ \dot{\phi}\cos\phi & 0 & -\dot{\phi}\sin\phi \end{bmatrix} \qquad 9\text{-}25$$

$$\left[\dot{R}_3\right] = \begin{bmatrix} -\dot{\psi}\sin\psi & \dot{\psi}\cos\psi & 0 \\ -\dot{\psi}\cos\psi & -\dot{\psi}\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad 9\text{-}26$$

Now if we let t approach t_1

$$\lim_{t \to t_1} \left[\dot{a}_{ij}\ (t)\right] = \lim_{t \to t_1} \left[\dot{b}_{ij}\ (t)\right] \left[a_{ij}\ (t_1)\right]$$

$$\lim_{t \to t_1} \left[\dot{b}_{ij}\ (t)\right] = \lim_{t \to t_1} \left[\left[\dot{R}_1\right] \left[R_2\right] \left[R_3\right] + \left[R_1\right] \left[\dot{R}_2\right] \left[R_3\right] + \left[R_1\right] \left[R_2\right] \left[\dot{R}_3\right] \right]. \qquad 9\text{-}27$$

Since

$$\lim_{t \to t_1} \phi(t) = \lim_{t \to t_1} \theta(t) = \lim_{t \to t_1} \psi(t) = 0 \qquad 9\text{-}28$$

then using (9-28) in (9-8), (9-9) and (9-10)

$$\lim_{t \to t_1} [R_1(t)] = \lim_{t \to t_1} [R_2(t)] = \lim_{t \to t_1} [R_3(t)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 9\text{-}29$$

and using (9-28) in (9-24), (9-25) and (9-26)

$$\lim_{t \to t_1} \left[\dot{R}_1(t)\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dot{\theta} \\ 0 & -\dot{\theta} & 0 \end{bmatrix} \qquad 9\text{-}30$$

$$\lim_{t \to t_1} \left[\dot{R}_2(t)\right] = \begin{bmatrix} 0 & 0 & -\dot{\phi} \\ 0 & 0 & 0 \\ \dot{\phi} & 0 & 0 \end{bmatrix} \qquad 9\text{-}31$$

$$\lim_{t \to t_1} \left[\dot{R}_3(t)\right] = \begin{bmatrix} 0 & \dot{\psi} & 0 \\ -\dot{\psi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad 9\text{-}32$$

Substituting (9-29) through (9-31) into (9-27) yields

$$\lim_{t \to t_1} \left[b_{ij}(t)\right] = \begin{bmatrix} 0 & \dot{\psi} & -\dot{\phi} \\ -\dot{\psi} & 0 & \dot{\theta} \\ \dot{\phi} & -\dot{\theta} & 0 \end{bmatrix} \qquad 9\text{-}33$$

and

$$\lim_{t \to t_1} \left[\dot{a}_{ij}(t)\right] = \begin{bmatrix} 0 & \dot{\psi} & -\dot{\phi} \\ -\dot{\psi} & 0 & \dot{\theta} \\ \dot{\phi} & -\dot{\theta} & 0 \end{bmatrix} \left[a_{ij}(t_1)\right]. \qquad 9\text{-}34$$

t_1 is an arbitrarily chosen time and (9-34) is true for any choice of t_1, so (9-34) is true in general.

$$\left[\dot{a}_{ij}\right] = \begin{bmatrix} 0 & \dot{\psi} & -\dot{\phi} \\ -\dot{\psi} & 0 & \dot{\theta} \\ \dot{\phi} & -\dot{\theta} & 0 \end{bmatrix} \left[a_{ij}\right]. \qquad 9\text{-}35$$

From the definitions of ψ, θ and ϕ; $\dot{\theta}$ is the rate of rotation of the s_i system about the s_1 axis; $\dot{\phi}$, the rate about the s_2 axis; $\dot{\psi}$, the rate about the s_3 axis. In other words, at the instant $t = t_1$, $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$ are equal to the components of the angular velocity vector $\vec{\omega}$ resolved in the s_1, s_2, s_3 system:

$$\vec{\omega}^s = \omega_1^s \vec{s}_1 + \omega_2^s \vec{s}_2 + \omega_3^s \vec{s}_3 \qquad 9\text{-}36$$

$$\vec{\omega}^s = \dot{\theta} \vec{s}_1 + \dot{\phi} \vec{s}_2 + \psi \vec{s}_3. \qquad 9\text{-}37$$

(9-35) can be rewritten:

$$\left[\dot{a}_{ij}\right] = \left[\omega_i^s\right] \left[a_{ij}\right] \qquad 9\text{-}38$$

where the symbol $\left[\omega_i^s\right]$ means

$$\left[\omega_i^s\right] = \begin{bmatrix} 0 & \omega_3^s & -\omega_2^s \\ -\omega_3^s & 0 & \omega_1^s \\ \omega_2^s & -\omega_1^s & 0 \end{bmatrix}. \qquad 9\text{-}39$$

Using (9-18) and (9-17) the last term in equation (9-18) can be written

$$\left[\dot{a}_{ij}\right] \vec{V}^r = \left[\omega_i^s\right] \left[a_{ij}\right] \vec{V}^r$$

$$= \left[\omega_i^s\right] \vec{V}^s. \qquad 9\text{-}40$$

Then finally, expansion of (9-40) will show:

$$\left[\omega_i^s\right] \vec{V}^s = -\vec{\omega}^s \times \vec{V}^s$$

so that

$$\dot{\vec{V}}^s = \left[a_{ij}\right] \dot{\vec{V}}^r - \vec{\omega}^s \times \vec{V}^s. \qquad 9\text{-}19$$

Another expression for V^s is obtained by multiplying (9-38) on the left by

$$\left[a_{ij}\right] \left[a_{ji}\right] = \left[1\right]$$

to get

$$\left[1\right] \left[\dot{a}_{ij}\right] = \left[\left[a_{ij}\right] \left[a_{ji}\right]\right] \left[\omega_i^s\right] \left[a_{ij}\right]$$

$$\left[\dot{a}_{ij}\right] = \left[a_{ij}\right] \left[\left[a_{ji}\right] \left[\omega_i^s\right] \left[a_{ij}\right]\right]$$

but, as can be verified by performing the multiplications,*

$$\left[a_{ji}\right] \left[\omega_i^s\right] \left[a_{ij}\right] = \left[\omega_i^r\right]$$

*Reference 6, p. 105.

so that

$$\left[\dot{a}_{ij}\right] = \left[a_{ij}\right] \quad \left[\omega_i^{\ r}\right]. \qquad \text{9-41}$$

Using (9-41) in (9-18) yields

$$\dot{\vec{V}}^s = \left[a_{ij}\right] \dot{\vec{V}}^r + \left[a_{ij}\right] \quad \left[\omega_i^{\ r}\right] \vec{V}^r$$

$$\dot{\vec{V}}^s = \left[a_{ij}\right] \left[\dot{\vec{V}}^r - \vec{\omega}^r \times \vec{V}^r\right]. \qquad \text{9-20}$$

The above derivation yields the same results as the classical approach. The advantage of (9-19) and (9-20) over the operator definition* is the inclusion of the rotation matrix in the equations.

<u>9.5.2</u> Taking the derivative of (9-18) yields

$$\ddot{\vec{V}}^s = \left[a_{ij}\right] \ddot{\vec{V}}^r + 2\left[\dot{a}_{ij}\right] \dot{\vec{V}}^r + \left[\ddot{a}_{ij}\right] \vec{V}^r \qquad \text{9-42}$$

and from (9-38)

$$\left[\ddot{a}_{ij}\right] = \frac{d}{dt} \left[\omega_i^{\ s}\right] \quad \left[a_{ij}\right]$$

$$= \left[\dot{\omega}_i^{\ s}\right] \quad \left[a_{ij}\right] + \left[\omega_i^{\ s}\right] \quad \left[\dot{a}_{ij}\right]. \qquad \text{9-43}$$

From (9-38), the second term of (9-43) is

$$\left[\omega_i^{\ s}\right] \left[\dot{a}_{ij}\right] = \left[\omega_i^{\ s}\right] \quad \left[\omega_i^{\ s}\right] \quad \left[a_{ij}\right]$$

so that (9-42) becomes

$$\ddot{\vec{V}}^s = \left[a_{ij}\right] \ddot{\vec{V}}^r + 2 \left[\omega_i^{\ s}\right] \left[a_{ij}\right] \dot{\vec{V}}^r$$

$$+ \left[\omega_i^{\ s}\right] \quad \left[\omega_i^{\ s}\right] \left[a_{ij}\right] \vec{V}^r + \left[\dot{\omega}_i^{\ s}\right] \left[a_{ij}\right] \vec{V}^r. \qquad \text{9-44}$$

*Reference 6, p. 133.

From (9-18)

$$\left[a_{ij}\right] \dot{\vec{V}}^{r} = \dot{\vec{V}}^{s} - \left[a_{ij}\right] \vec{V}^{r}$$

$$= \dot{\vec{V}}^{s} - \left[\omega_i{}^{s}\right] \vec{V}^{s} . \qquad 9\text{-}45$$

Substituting (9-45) into (9-44) yields

$$\ddot{\vec{V}}^{s} = \left[a_{ij}\right] \ddot{\vec{V}}^{r} + 2 \left[\omega_i{}^{s}\right] \dot{\vec{V}}^{s} - \left[\omega_i{}^{s}\right] \left[\omega_i{}^{s}\right] \vec{V}^{s} + \left[\dot{\omega}_i{}^{s}\right] \vec{V}^{s}$$

$$= \left[a_{ij}\right] \ddot{\vec{V}}^{r} - 2\, \vec{\omega}^{s} \times \dot{\vec{V}}^{s} - \vec{\omega}^{s} \times \vec{\omega}^{s} \times \vec{V}^{s} - \dot{\vec{\omega}}^{s} \times \vec{V}^{s} . \qquad 9\text{-}46$$

10. Equations

10.1 Tower Phase Equations

Tower phase equations are used in Programs 1 through 4, if the initial z_m^{ℓ} is less than z_T.

10.1.1 Variables of the Differential Equations

$$x_1 = \dot{x}_m^{\ell}$$

$$x_2 = x_m^{\ell}$$

$$x_3 = \dot{z}_m^{\ell}$$

$$x_4 = z_m^{\ell}$$

10.1.2 Initial Conditions

$$x_i \Big|_o = 0$$

10.1.3 Differential Equations

$$\dot{x}_1 = (\left|T\right| - \left|D\right| - M \left|E\right| \cos \theta_o) \sin \theta_o / M$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = (\left|T\right| - \left|D\right| - M \left|E\right| \cos \theta_o) \cos \theta_o / M$$

$$\dot{x}_4 = x_3$$

10.1.4 Exit Conditions

Exit tower phase when

$$z_m^{\ell} = z^T$$

by an altitude discontinuity check. Exit to the atmospheric equations.

10.1.5 Functional Equations

$\lvert T \rvert$	from 5-1
$\lvert D \rvert$	from 6-5
M	from mass table (Section 5)
$\lvert E \rvert$	from 4-9
Θ_o	trajectory parameter

10.2 Two Dimensional Particle Atmospheric Equations

Two Dimensional Particle Atmospheric equations are used:

(a) in Program 1 starting from tower exit,

(b) in Programs 2, 3, and 4 after angle of attack oscillations damp out,

(c) in Programs 1 through 4, from re-entry to impact.

10.2.1 Variables of the Differential Equations

$$x_1 = \dot{x}_m^{\ell}$$

$$x_2 = x_m^{\ell}$$

$$x_3 = \dot{z}_m^{\ell}$$

$$x_4 = z_m^{\ell}$$

10.2.2 Initial Conditions

(a) For Program 1, starting in tower, initial conditions are tower exit conditions.

(b) For Program 1, starting above tower, initial conditions are read in on cards.

(c) For Programs 2, 3 and 4, starting at the switch from rigid body equations, initial conditions are obtained from the rigid body equations.

(d) For Programs 1 through 4, on re-entry, initial conditions are obtained from the vacuum equations.

10.2.3 Differential Equations

$$\dot{x}_1 = (|T| - |D|) \sin \Theta/M$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = (|T| - |D|) \cos \Theta/M - |E|$$

$$\dot{x}_4 = x_3$$

10.2.4 Exit Conditions

(a) During ascending flight, exit when $z_m^{\ell} + H_L = 300{,}000$ ft. by a discontinuity check and after burnout of last stage. Exit to vacuum differential equations or to closed form equations controlled by program switch.

(b) During descending flight, exit when $z_m^{\ell} = 0$ by a discontinuity check. Exit to next trajectory, with $\Theta_o + \Delta\Theta$ replacing Θ_o, or to program halt if Θ_o is Θ_{max}.

10.2.5 Functional Equations

$|T|$ from 5-1

$|D|$ from 6-5

M from mass table

$|E|$ from 4-9

$$\sin\Theta = \dot{x}_m^{\ell}/V$$

$$\cos\Theta = \dot{z}_m^{\ell}/V$$

$$V = \sqrt{(\dot{x}_m^{\ell})^2 + (\dot{z}_m^{\ell})^2}$$

10.3 Vacuum Equations

Vacuum equations are used in Programs 1 through 4, when $(z_m^{\ell} + z_H) >$ 300,000 feet, and after thrust terminates.

10.3.1 Variables of the Differential Equations

$$x_1 = \dot{x}_m^{\ell}$$

$$x_2 = x_m^{\ell}$$

$$x_3 = \dot{z}_m^{\ell}$$

$$x_4 = z_m^{\ell}$$

10.3.2 Initial Conditions

Initial conditions are obtained from last point of atmospheric equations.

10.3.3 Differential Equations

$$\dot{x}_1 = 0$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = -\left|E\right|$$

$$\dot{x}_4 = x_3$$

10.3.4 Exit Conditions

Exit when $(z_m^{\ell} + H_L) = 300{,}000$ feet on the descent by a discontinuity check.

10.3.5 Functional Equations

$\left|E\right|$ from 4-9

10.4 Closed Form Vacuum Equations

Closed form vacuum equations are a program option to be used instead of vacuum differential equations.

10.4.1 Initial Conditions

$$\begin{pmatrix} \dot{x}_v \\ x_v \\ \dot{z}_v \\ z_v \end{pmatrix} = \left.\begin{pmatrix} \dot{x}_m^{\ell} \\ x_m^{\ell} \\ \dot{z}_m^{\ell} \\ z_m^{\ell} \end{pmatrix}\right|_{\text{entry to vacuum}}$$

10.4.2 Apogee Equations

$$\begin{pmatrix} \dot{x}_p \\ x_p \\ \dot{z}_p \\ z_p \\ t_v \end{pmatrix} = \left.\begin{pmatrix} \dot{x}_m^{\ell} \\ x_m^{\ell} \\ \dot{z}_m^{\ell} \\ z_m^{\ell} \\ t \end{pmatrix}\right|_{\text{peak}}$$

$$z_p = z_v + \left[\dot{z}_v^2 (R_o + z_v)^2\right] / \left[2g_o R_o^2 - \dot{z}_v^2 (R_o + z_v)\right]$$

$$t_p = t_v + \sqrt{(z_p - z_v)(R_o + z_v)(R_o + z_p)}$$

$$+\left(\sqrt{(R_o + z_p)^{3/2}\, g_o\, R_o^2}\right)\left(\tan^{-1}\sqrt{(z_p - z_v)/(R_o + z_v)}\right)$$

$$x_p = x_v + \dot{x}_v\,(t_p - t_v)$$

$$\dot{x}_p = \dot{x}_v$$

$$\dot{z}_p = 0$$

10.4.3 Re-entry Equations

$$\begin{pmatrix} \dot{x}_r \\ x_r \\ \dot{z}_r \\ z_r \\ t_r \end{pmatrix} = \left.\begin{pmatrix} \dot{x}_m\ell \\ x_m\ell \\ \dot{z}_m\ell \\ z_m\ell \\ t \end{pmatrix}\right|_{\text{re-entry}}$$

$$\dot{x}_r = \dot{x}_v$$

$$x_r = x_v + 2\dot{x}_v\,(t_p - t_v)$$

$$\dot{z}_r = \dot{z}_v$$

$$z_r = z_v$$

10.5 Rigid Body Equations

Rigid Body equations are used in Programs 2 through 4, beginning at tower exit.

10.5.1 Variables of the Differential Equations

$$x_1 = V_1^{\ell}$$

$$x_2 = V_3^{\ell}$$

$$x_3 = \dot{\beta}$$

$$x_4 = \beta$$

$$x_5 = x_m^{\ell}$$

$$x_6 = z_m^{\ell}$$

$$x_7 = \phi \text{ (Program 5 only)}$$

10.5.2 Initial Conditions

$$x_1 = \left| \dot{\vec{R}}_m^{\ell} \right| \cos \beta_o$$

$$x_2 = \left| \dot{\vec{R}}_m^{\ell} \right| \sin \beta_o$$

$$x_3 = \dot{\beta}_o$$

$$x_4 = \beta_o$$

$$x_5 = x_m^{\ell}$$

$$x_6 = z_m^{\ell}$$

$$x_7 = \phi_o \text{ (Program 5 only)}$$

$$\left.\begin{array}{l} \dot{\beta}_o \\ \beta_o \\ \phi_o \end{array}\right\} \quad \text{trajectory parameters from input data}$$

$$\left.\begin{array}{l} x_m^{\ell} \\ z_m^{\ell} \\ \dot{\vec{R}}_m^{\ell} \end{array}\right\} \quad \text{from tower exit conditions}$$

10.5.3 Differential Equations

$$x_1 = \frac{1}{M}\left(T_1^{\,b} + A_1^{\,b}\right) + E_1^{\,b} - x_3\, x_2$$

$$x_2 = \frac{1}{M}\left(T_3^{\,b} + A_3^{\,b}\right) + E_3^{\,b} + x_3\, x_1$$

$$\dot{x}_3 = \frac{1}{I}\left(\left|\vec{M}_T^{\,b}\right|^* + \left|\vec{M}_j^{\,b}\right| + \left|\vec{M}_{L,D}^{\,b}\right| + \left|\vec{M}_{FM}^{\,b}\right|^* + \left|\vec{M}_{\dot{\beta}}^{\,b}\right| - \dot{I}\, x_3\right)$$

$$\dot{x}_4 = x_3$$

$$\dot{x}_5 = x_1 \sin\beta - x_2 \cos\beta$$

$$\dot{x}_6 = x_1 \cos\beta + x_2 \sin\beta$$

$$\left.\begin{array}{l} \dot{x}_7 = K_R \left|V_R\right| \\ \qquad \text{or} \\ \dot{x}_7 = \dot{\phi}\,(t) \text{ from table} \end{array}\right\} \quad \text{program option in Program 4}$$

10.5.4 Exit Conditions

(a) Exit to particle when angle of attack has been less than a pre-set ϵ for ten successive iterations.

(b) Exit to particle when a pre-set altitude is reached.

(c) Exit to program halt at impact $z_m^{\ell} = 0$.

*Program 4 only.

10.5.5 Functional Equations

$\vec{T}^b$	5-2 or 5-3
$\vec{A}^b$	6-9 or 6-13
$\vec{E}^b$	4-9 and 4-16
$\vec{M}_T{}^b$	5-6 (Program 4 only)
$\vec{M}_j{}^b$	5-7
$\vec{M}_{L,P}{}^b$	6-14
$\vec{M}_{FM}{}^b$	6-15
$\vec{M}_{\dot{\beta}}{}^b$	6-17

10.6 Three Dimensional Particle Equations

Three Dimensional Particle equations are used throughout Program 5.

10.6.1 Variables of the Differential Equations

$$x_1 = \dot{x}_m{}^f$$

$$x_2 = x_m{}^f$$

$$x_3 = \dot{y}_m{}^f$$

$$x_4 = y_m{}^f$$

$$x_5 = \dot{z}_m{}^f$$

$$x_6 = z_m{}^f$$

10.6.2 Initial Conditions

(a) For tower phase

$$x_i\Big|_o = 0$$

(b) For atmosphere phase

$$x_i\Big|_o = x_{i\,\text{tower exit}}$$

(c) For vacuum phase

$$x_i\Big|_o = x_i\Big|_{H_M = H_V}$$

H_V = 300,000 feet or $H_V = H_M$ at thrust termination whichever is greater

10.6.3 Differential Equations

(a) Tower phase

$$\dot{x}_1 = \left[\left[|T| - |D| + M\,|E|\cos\theta\right]\sin\theta\cos\psi\right]/M$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = \left[\left[|T| - |D| + M\,|E|\cos\theta\right]\sin\theta\sin\psi\right]/M$$

$$\dot{x}_4 = x_3$$

$$\dot{x}_5 = \left[\left[|T| - |D| + M\,|E|\cos\theta\right]\cos\theta\right]/M$$

$$\dot{x}_6 = x_5$$

(b) Atmosphere phase

$$\dot{x}_1 = \left[\left[|T| - |D| \right] \sin \Theta \cos \Psi \right] / M + E_1{}^f$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = \left[\left[|T| - |D| \right] \sin \Theta \sin \Psi \right] / M + E_2{}^f$$

$$\dot{x}_4 = x_3$$

$$\dot{x}_5 = \left[\left[|T| - |D| \right] \cos \Theta \right] / M + E_3{}^f$$

$$\dot{x}_6 = x_5$$

(c) Vacuum phase

$$\dot{x}_1 = E_1{}^f$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = E_2{}^f$$

$$\dot{x}_4 = x_3$$

$$\dot{x}_5 = E_3{}^f$$

$$\dot{x}_6 = x_5$$

10.6.4 Exit Conditions

Exit Conditions are identical to those of Program 1.

10.6.5 Functional Equations

$|T|$ from 5-1

$\lvert D \rvert$	from 6-5. Note: H_M is used for atmosphere table lookups
θ_o	Tower setting
θ	$\tan^{-1}\left[\sqrt{(\dot{x}_m{}^f)^2 \div (\dot{y}_m{}^f)^2} \;/V\right]$
ψ	$\tan^{-1}(\dot{x}_m{}^f/\dot{y}_m{}^f)$
$\vec{E}^f$	7-14

APPENDIX I

GLOSSARY

The section listing below gives sections in which the symbols are defined in context.

For explanation of vector subscripting and superscripting conventions, see Section 9.

Symbol	Section	Definition
$\vec{A}$	4,6	Aerodynamic Force
A_E	5	Rocket motor exit area
a	7,8	Semi-major axis of the earth
a_{ij}	9	Element in i^{th} row and j^{th} column of $[a_{ij}]$
$[a_{ij}]$	4	Direction cosine matrix transforming from launcher system to body system
$[a_{ij}]$	9	General direction cosine matrix
b_i	4-7	Body coordinate system axes
b-superscript		Indicates vector resolved in body system
b-subscript	5	Rocket motor burnout
$[b_{ij}]$	8	Matrix of transformation from ℓ_i system to t_i system
$[b_{ij}]$	9	General direction cosine matrix
$\vec{C}$	4,7	Centrifugal force vector
$\vec{\mathcal{C}}$	8	Coriolis force vector

Symbol	Section	Definition
C	8	Center of the earth
c_i	8	Axes of earth fixed coordinate system with origin at the earth's center
c-superscript		Indicates vector resolved in c_i system
$[c_{ij}]$		General direction cosine matrix
C_D	6	Drag coefficient
C_L	6	Lift coefficient
$C_{L\alpha}$	6	Slope of C_L vs. α, at $\alpha = 0$
$C_{L\alpha\,F}$	6	Slope of C_L for fins alone vs. α, at $\alpha = 0$
c.g.	4-7	Center of gravity of rocket
c.p.	6	Center of pressure of rocket
c.p.f.	6	Center of pressure of fins
$\vec{D}$	6	Drag force vector
d^2	6	Reference area for aerodynamic coefficients
$[d_{ij}]$	9	General direction cosine matrix
$\vec{E}$	4,7	Net acceleration vector caused by gravitation and rotation of the earth
$\vec{F}$	4	Force vector
f_i ( )	2	Function of variables in parentheses
f_i	4,8	Axes of system aligned with firing azimuth
f-superscript	4,8	Vector resolved in f_i system
$\vec{G}$	4,7	Gravitational acceleration vector
g_o	4	Magnitude of $\vec{G}$ at a reference point
g_i	4,8	Ground fixed coordinate axes
g-superscript		Indicates vector resolved in g_i system

Symbol	Section	Definition
H_L	8	Geodetic height of launcher
$\vec{H}_L$	8	Vector from surface of reference ellipsoid to L, in plumb line direction
H_M	8	Geodetic height of rocket
$\vec{H}_M$	8	Vector from surface of reference ellipsoid to rocket, in plumb line direction
H_T	8	Geodetic height of target
h	6,8	Height of rocket
I, I_2	4	Transverse moment of inertia of rocket
I-subscript	5	Rocket motor impulse
I-superscript	4	Vector resolved in inertial system
J	5	Total impulse of rocket motor
x_{ij}	2	Increments of variable in Runge-Kutta integration
k	5	Constant of proportionality between thrust and mass flow rate
k_{G_i}	7	$i = 1, \ldots, 6$ constants of gravitational potential
k_i	8	$i = 1, 2, 3$ translation constants for transforming from ℓ_i to t_i system
L	8	As a point: Launcher location As a subscript: Pertaining to the launcher
L'	8	Foot of perpendicular from L to reference ellipsoid
L	8	Geodetic latitude

Symbol	Section	Definition
$L_{subscript}$	8	Geodetic latitude of point indicated by subscript
$\vec{L}$	6	Lift force vector
$\vec{L}_{FM}$	6	Lift force caused by fin malalignment
$\vec{L}_{T}$	6	Lift force caused by tail surface alone
$\mathcal{L}$	8	Geocentric latitude
$\mathcal{L}$-subscript	8	Geocentric latitude of point indicated by subscript
ℓ_i	4,8	Axes of launcher based system
ℓ-superscript	8	Vector resolved in ℓ_i system
ℓ-subscript	8	Vector directed to launcher
M	8	As a point: Location of rocket As a subscript: Pertaining to rocket location
M	4	Mass of rocket
$\vec{M}$	4	Moment vector
$\vec{M}_T$	5	Moment caused by thrust malalignment
$\overline{M}_j$	5	Moment caused by jet damping
$\vec{M}_{L,D}$	6	Moment caused by lift and drag
$\vec{M}_{FM}$	6	Moment caused by fin malalignment
$\vec{M}_{\dot{\beta}}$	6	Moment caused by aerodynamic damping
$\mathcal{m}$	6	Mach number
n	2	Number of first order differential equations in set
o-subscript		Initial condition
P	8.4	A point on the surface of the reference ellipsoid

Symbol	Section	Definition
P-subscript		Pertaining to P, above
P	6	Atmospheric pressure
P_A	5,6	Ambient atmospheric pressure: pressure at specified location
P_S	5,6	Sea level atmospheric pressure
P-subscript	10	Condition at peak of trajectory
q	6	Dynamic pressure
$\vec{R}$	All	Position vector
$\dot{\vec{R}}$	All	First derivative of $\vec{R}$
$\ddot{\vec{R}}$	All	Second derivative of $\vec{R}$
$\vec{R}^{\text{superscript}}_{\text{subscript}}$	All	Vector from origin of system indicated by superscript to point indicated by subscript
R_o	4	Distance from center of earth to point chosen for reference gravitational magnitude g_o
$\vec{R}_P$	6	Vector from c.g. to c.p.
$\vec{R}_{PF}$	6	Vector from c.g. to c.p.f.
$\vec{R}_T$	5	Vector from c.g. to center motor exit plane
$[R_1]$, $[R_2]$, $[R_3]$	All	Elementary rotations about the "1", "2" and "3" axes
r_i	9	Axes of a general coordinate system
r-subscript	10	Re-entry condition
s_i	9	Axes of a general coordinate system
t	All	Time
$\vec{T}$	4,5	Thrust vector

Symbol	Section	Definition
T	8	As a point: Target location As a subscript: Pertaining to target location
T'	8	Foot of perpendicular from target location to reference ellipsoid
t_i	8	Axes of target based system
$\vec{V}$	9	General vector
$\vec{V}^b$	4	Velocity of rocket as defined in 4.3
$\vec{V}_E$	5	Velocity of rocket exhaust with respect to rocket
v-subscript	10	Conditions at entry to vacuum conditions
$\vec{W}$	6	Wind velocity
w_i	6	Axes of wind oriented system
x_i, x_j	2	General variable of differential equations
$x^{\text{superscript}}_{\text{subscript}}$ $y^{\text{superscript}}_{\text{subscript}}$ $z^{\text{superscript}}_{\text{subscript}}$	All	Coordinates of point designated by subscript in system designated by superscript: components of $\vec{R}^{\text{superscript}}_{\text{subscript}}$
X_T	6	Distance from c.g. to center of pressure of tail
X_N	6	Distance from c.g. to center of pressure of nose
α	6	Angle of attack
α_F	6	Angle of attack of fins
α_T	6	Angle of attack of tail section
β	6	Pitch angle

Symbol	Section	Definition
$\dot{\beta}$	6	Pitch rate
γ	6	Ratio of specific heats for the atmosphere
Δt	2	Time interval for numerical integration
$\Delta \Lambda$	8	Difference between longitude of launcher and rocket
$\Delta \Theta$	10	Increment in launcher setting from one trajectory to the next
δF	6	Fin malalignment angle
δL	8	Difference between geodetic and geocentric latitudes of a point on the surface of the ellipsoid
$\delta \mathscr{L}$	8	Difference between the geocentric latitudes of two points
ϵ	5	Thrust malalignment angle
Θ	10	Angle between velocity vector and the ℓ_3 (vertical) axis
Λ	8	Longitude
Σ	All	Summation
$\sum_i$	All	Summation over the index i
$\vec{\rho}$ -subscript	8	Vector from center of earth to point indicated by subscript
ρ	8	Radius of earth
Φ	7	Gravitational potential function
ϕ	5,6	Roll angle of rocket

Symbol	Section	Definition
ϕ_i	2	General function
ψ	10	Heading angle of rocket in f_i system
$\vec{\omega}$	All	Angular rotation vector
ω_i	All	Components of $\vec{\omega}$
$[\omega_i]$	All	Matrix such that the element in the j^{th} row and k^{th} column is $\omega_i \; \delta_{ijk}$, where δ_{ijk} is the generalized Kroneker delta

APPENDIX II

REQUIRED INPUT PARAMETERS

Launch Point Parameters

Symbol	Units	Description	Program 1	2	3	4	5
θ_o, $\Delta\theta$, θ_L	Degrees	Range and increment of launch angle, measured from vertical	X	X	X	X	X
FAZ	Degrees	Firing azimuth, measured positive clockwise from north					X
LDLA	Degrees	Launch point geodetic latitude, measured positive north from equator					X
LLO	Degrees	Launch point longitude, measured positive west from Greenwich					X
LDH	Feet	Launch point geodetic heighth above sea level					X
ZL	Feet	Launch point heighth above sea level	X	X	X	X	
ZT	Feet	Launcher length	X	X	X	X	X
g_o	Feet/sec.^{2}	Acceleration due to gravity	X	X	X	X	X
TDLA	Degrees	Target geodetic latitude					X
TLO	Degrees	Target longitude					X
TDH	Feet	Target geodetic height					X

Initial Position and Velocity Parameters

Symbol	Units	Description	Program 1	2	3	4	5
T_O	Seconds	Initial time	X	X	X	X	X
X	Feet	Initial range from launch point, measured along firing azimuth line	X	X	X	X	X
Y	Feet	Initial cross range					X
Z	Feet	Initial altitude, above launch point	X	X	X	X	X
U	Feet/sec.	Initial velocity component along longitudinal axis of rocket		X	X	X	
W	Feet/sec.	Initial velocity component normal to longitudinal axis of rocket		X	X	X	
$\dot{X}$	Feet/sec.	Initial horizontal velocity component	X			X	
$\dot{Z}$	Feet/sec.	Initial vertical velocity component	X			X	
V_O	Feet/sec.	Initial velocity					X
Phi	Degrees	Initial heading, measured positive clockwise from FAZ line					X
$\dot{\beta}$	Degrees/sec.	Initial pitch rate, measured positive from vertical		X	X	X	
β	Degrees	Initial pitch angle, measured from vertical		X	X	X	
Θ	Degrees	Initial flight path angle, measured positive from vertical	X	X			X

Stage (Phase) Parameters

TS	Seconds	Time of separation	X	X	X	X	X
T_I	Seconds	Time of ignition	X	X	X	X	X

Symbol	Units	Description	Program 1	2	3	4	5
T_{BO}	Seconds	Time of burnout	X	X	X	X	X
CT	Feet	Center of thrust, measured from base of rocket to throat of nozzle		X	X	X	
CPD	Feet	Damping center of pressure, measured from base of rocket		X	X	X	
CPF	Feet	Fin center of pressure, measured from base of rocket				X	
N/CLSF*		Fin lift factor (% of total lift)				X	
KR**	Radians/ft.	Roll rate = (KR) × velocity				X	
A_E	$(\text{Inch})^2$	Exit area	X	X	X	X	X
PL	Pounds	Payload weight	X	X	X	X	X
PL Sep.	Seconds	Time of payload separation	X	X	X	X	X
Ref. Area	$(\text{Inches})^2$	Reference area for aerodynamic coefficients					

Table Parameters for Each Stage or Flight Phase

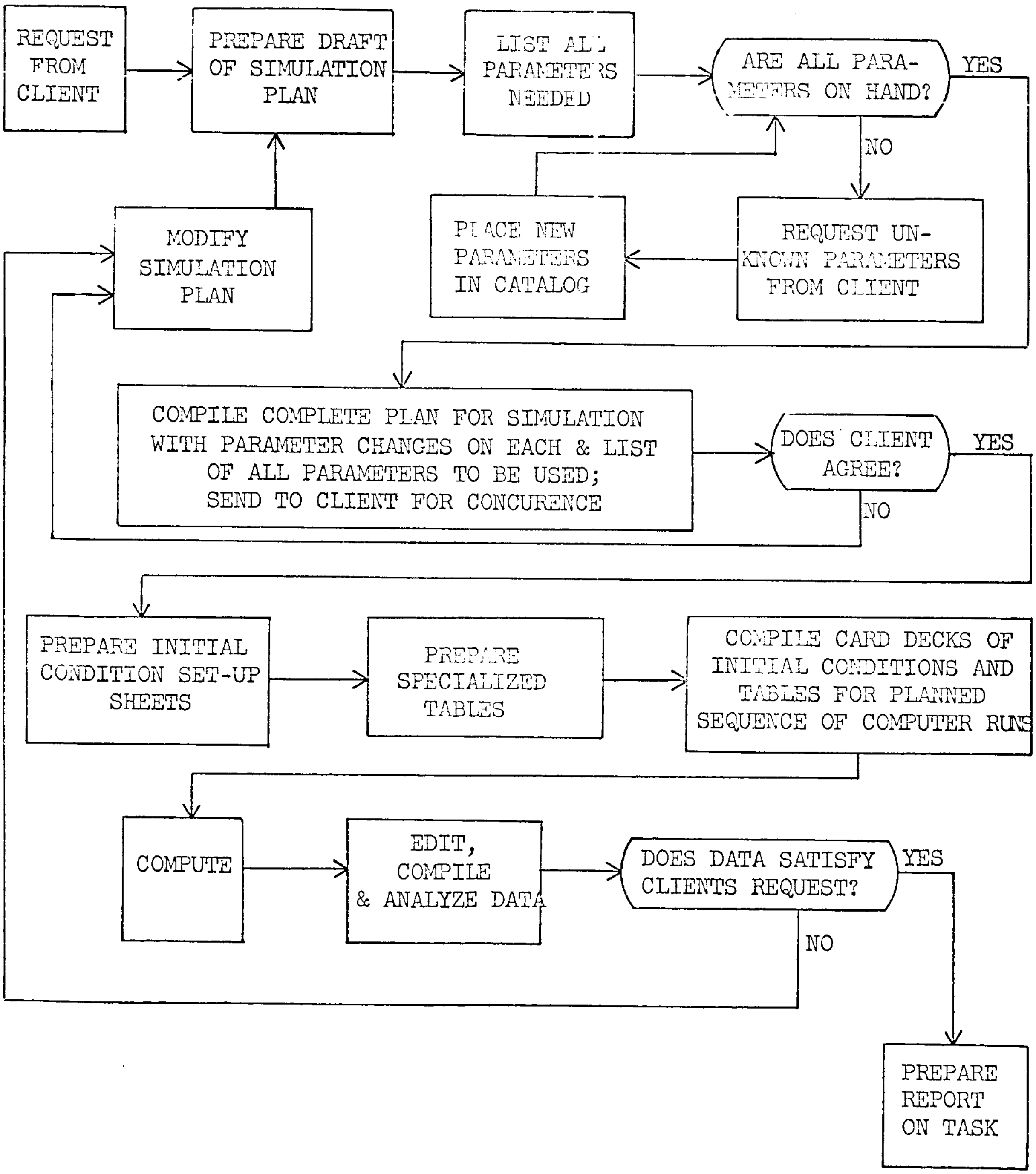
			1	2	3	4	5
Thrust vs. Time	Pounds		X	X	X	X	X
Weight vs. Time	Pounds		X	X	X	X	X
Drag Coefficient vs. Mach No.			X	X	X	X	X
Lift Coefficient Slope vs. Mach No.	$(\text{Radian})^{-1}$			X	X	X	
Center of Pressure vs. Mach No.	Feet			X	X	X	
Moment of Inertia vs. Time	Slug-ft.^{2}			X	X	X	
Center of Gravity vs. Time	Feet			X	X	X	

* N/CLSF can be substituted by a table of Fin Lift Coefficient Slope vs. Mach No.
**KR can be substituted by a table of Roll Rate vs. Time.

APPENDIX III

OPERATIONAL FLOW PROCEDURE FOR

COMPUTER SIMULATION TASKS



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